## 1.4 Representation Theory

In the previous section, we looked around the group manifold. The theory of Lie groups is a very interesting subject but it is only meaningful through the representation of Lie groups (and algebra) in most case of the physics.
In this section, I will focus on the representation of semisimple Lie algebra in most case since they are well-classified and the most important Lie algebras in physics are usually semisimple (We will show the definition of it soon. Don't panic!). However, the Poincaré group, another important Lie group in the physics, is not a semisimple Lie group. For this reason, I will also discuss about representation of Poincaré algebra. Finally, I will cover the Young diagram of $\mathfrak{so}(N)$ representation which give intuitions on the symmetry of the system (e.g., mixed symmetric field).
This section is primarily based on Dr. Thomas Basile's lecture on the representation of semisimple Lie algebras in 2019 at KHU, and also I referred other literature [3–6].

### 1.4.1 Structure of Representation Theory

Before discussing the construction of a representation of a Lie algebra, it is necessary to address the representations itself as our primary subject. We begin with the definition.

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**Definition** | **Representation of a Lie Algebra**

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Let $\mathfrak{g}$ is a Lie algebra, and $V$ is a $\mathbb{C}$-vector space. Then, a **representation** $\rho$ of $\mathfrak{g}$ on $V$ is an endomorphism

$$\begin{aligned} \rho\,:\;\mathfrak{g} &\to \mathrm{End}(V) \\ X &\mapsto \rho(X) \quad , \end{aligned} \tag{1.4.1}$$

which should preserves the Lie bracket as a commutator relation

$$\rho(X)\rho(Y) - \rho(Y)\rho(X) = \rho([\![X, Y]\!]_{\mathfrak{g}})\,, \tag{1.4.2}$$

for all $X, Y \in \mathfrak{g}$

Note that the complex vector space $V$, on which the representation $\rho$ actsm is called a $\mathfrak{g}$-module, and the dimension of the $\mathfrak{g}$-module is simply referred to as the dimension (or degree) of the representation. Sometimes, $V$ is referred to as a *representation* in the physics contexts, but in this section, I will distinguish two terminologies.
The first example is the trivial representation. If $\rho$ maps all $X \in \mathfrak{g}$ to $1$, then it might satisfy all definitions of a representation. In most cases, we demand for distinguishing different Lie algebra elements through the representation. This implies that only injective maps $\rho$, called as **faithful representations**, are interesting representations in the physics. In contrast, if the map $\rho$ is not injective, then it is referred as an unfaithful representation.
If $V$ contains an invariant subspace $W$, such that $\rho(X)w \in W$ , $\forall X \in \mathfrak{g}$, for $w \in W$, then $\rho$ is called a reducible representation. Conversely, if $V$ contains no invariant subspace, then $\rho$ is called an **irreducible representation**. Our focus will be on faithful and irreducible representations unless otherwise specified.

One example of a representation —and of course, there are many!— is the Heisenberg algebra, denoted as $\mathfrak{heis}_3$. The Heisenberg algebra is spanned by three generators, $x$ , $y$ , and $z$, which satisfy the Lie algebra relation,

$$[x\,,z] = [y\,,z] = 0\,, \quad [x\,,y] = z\,. \tag{1.4.3}$$

We can consider a representation of $\mathfrak{heis}_3$ as follows:

$$\rho(x) = a\,, \quad \rho(y) = a^\dagger\,, \quad \rho(z) = 1\,, \tag{1.4.4}$$

where $a$ and $a^\dagger$ are the annihilation and creation operator, respectively, in quantum harmonic oscillator, and their multiplication is defined by the commutator relation,

$$[a\,,a^\dagger] = 1\,. \tag{1.4.5}$$

One important representation is the **adjoint representation**, denoted by $\mathrm{ad}_X$. The adjoint representation is defined as $\mathrm{ad}_X = [X\,,\cdot]$, so its module is $\mathfrak{g}$ itself, and the dimension of $\mathrm{ad}$ is same as that of $\mathfrak{g}$. Let $t_\alpha$ be the generator of a certain Lie algebra $\mathfrak{g}$, such that

$$[t_\alpha\,,\ t_\beta] = f_{\alpha\beta}{}^\gamma\, t_\gamma\,, \tag{1.4.6}$$

and consider the $\mathrm{ad}_{t_\alpha}$ action on another generator, namely $t_\beta$,

$$\mathrm{ad}_{t_\alpha}\, t_\beta = D(t_\alpha)_\beta{}^\gamma\, t_\gamma\,, \tag{1.4.7}$$

where $D$ is a linear map corresponding to the $ad_X$. Since the adjoint representation acts as a Lie algebra (1.4.6), we find

$$D(t_\alpha)_\beta{}^\gamma = f_{\alpha\beta}{}^\gamma\,. \tag{1.4.8}$$

Let us consider an example of Lie algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2) \simeq \mathfrak{sl}(2)$, which is spanned by the generators $\sigma_i$ satisfying

$$[\sigma_i\,,\sigma_j] = \epsilon_{ij}{}^k\,\sigma_k\,. \tag{1.4.9}$$

In the quantum mechanics, we have chosen a two-dimensional $\mathfrak{su}(2)$-module, denoted by $|\pm\rangle$, to find a two-dimensional representation, namely, the Pauli matrices. If we choose the $\mathfrak{su}(2)$ algebra as the module, then $(\mathrm{ad}_{\sigma_i})_j{}^k$ should be expressed by $\epsilon_{ij}{}^k$. For instance, $\mathrm{ad}_{\sigma_1}$ in the matrix form is given by

$$\mathrm{ad}_{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,. \tag{1.4.10}$$

Remark that adjoint representations play crucial roles in physics. as all gauge bosons are adjoint representation of their gauge groups. For instance, gluon fields are adjoint representation of $SU(3)$, which accounts for the existence of eight gluon fields.

Later, we will focus on the (semi)simple Lie algebras. To define the (semi)simplicity of a Lie algebra, it is necessary to define the **ideal** subalgebra

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**Definition** | **Ideal subalgebra**

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$\mathfrak{i} \subset \mathfrak{g}$ is called an ideal subalgebra if

$$[X\,,Y] \in \mathfrak{i} \tag{1.4.11}$$

for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{i}$.

One probably notice that the sets $\{0\}$ and $\mathfrak{g}$ must be ideal subalgebras of $\mathfrak{g}$. However, they are trivial, and a proper ideal is one that is not trivial. One simple example is algebras which consist of upper triangular matrices. Let $t(n)$ be a Lie algebra consisting of upper triangular $n \times n$ matrices with matrix commutator, and $\bar{t}(n)$ be a Lie algebras with *strictly* upper triangular $n \times n$ matrices, then one can show that $\bar{t}(n)$ is an ideal subalgebra of $t(n)$. An interesting feature of $\bar{t}(n)$ is that for any $A \in \bar{t}(n)$, $A^n$ is the zero matrix, indicating that $A$ is a nilpotent matrix. Related to this example, we define *nilpotent* and solvable Lie algebras.

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**Definition** | **Nilpotent & solvable Lie algebra**

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First, we define a ($\text{k}^{\underline{\text{th}}}$ ) derived series, denoted $\mathfrak{g}^{(k)}$. The derived series is iteratively defined as

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}\,,\mathfrak{g}^{(k-1)}]\,, \quad \mathfrak{g}^{(0)} = \mathfrak{g}\,. \tag{1.4.12}$$

Additionally, ($\text{k}^{\underline{\text{th}}}$ ) lower central series, denoted $\mathfrak{g}_{(k)}, isdefinedas$:

$$\mathfrak{g}_{(k)} = [\mathfrak{g}\,,\mathfrak{g}_{(k-1)}]\,, \quad \mathfrak{g}_{(0)} = \mathfrak{g}\,. \tag{1.4.13}$$

If $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$, the $\mathfrak{g}$ is called solvable.
Similarly, if $\mathfrak{g}_{(n)} = 0$ for some $n \in \mathbb{N}$, then $\mathfrak{g}$ is called nilpotent.

The Heisenberg algebra $\mathfrak{heis}_1$ provides a good intuition to us for solvability and nilpotency of the Lie algebra. Using the relation of $\mathfrak{heis}_1$ in (1.4.3), we find the derived series

$$\mathfrak{g}^{(1)} = \{z\}\,, \quad \mathfrak{g}^{(2)} = \{0\}\,, \dots\,, \tag{1.4.14}$$

and the lower central series,

$$\mathfrak{g}_{(1)} = \{0\}\,, \dots\,. \tag{1.4.15}$$

We find that $\mathfrak{heis}_1$ is both solvable and nilpotent. Actually, this is related to Engel's theorem and Lie's theorem. Here, we introduce statements of two theorems without proof.

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**Theorem** | **Engel's theorem**

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Any nilpotent Lie algebra is ad-nilpotent, that is, its adjoint representation is composed of nilpotent matrices.

**Theorem** | **Lie's theorem**

Let $\mathfrak{g}$ be a solvable Lie algebra and $(V\,,\rho)$ is a finite dimensional representation of $\mathfrak{g}$ the all $\rho(X)$ are upper triangular matrices in some basis.

Now, we define (semi)simpe Lie algebras:

**Definition** | **Simple/Semisimple Lie algebra**

$\mathfrak{g}$ is simple if it contains no proper ideals.
$\mathfrak{g}$ is semisimple if it has no abelian ideals.

An alternative version of definition of semisimple Lie algebra is as the direct sum of simple algebras. For example, $\mathfrak{so}(n)$ are simple Lie algebra unless $n = 4$, and $\mathfrak{so}(4)$ is a semisimple Lie algebra since it can be given as the direct sum of two $\mathfrak{so}(3)$.
Note that a semisimple algebra has no centre by definition [1], so the adjoint representation of a semisimple Lie algebra is faithful. But why semisimple Lie algebras are crucial? This is because both these Lie algebras and their representations are well-classified. Also, most of the important Lie algebras in physics are (semi)simple. One exception is the Poincaré algebra $\mathfrak{iso}(1,d-1)$. Even though it is one of the most important Lie algebras, it is not a semisimple one.
Another reason for the importance of semisimplicity is provided by **Levi-Malcev decomposition**. Any finite-dimensional Lie algebra $\mathfrak{g}$ can be decomposed into a semi-direct sum of a solvable piece, denoted $R$ (called radical) and a simple piece, denoted $S$. That is,

$$\mathfrak{g} = S \Subset_\sigma R\,, \tag{1.4.16}$$

where $\Subset$ is the semi-direct sum between $S$ and $R$, and $(R,\sigma)$ is a representation of $S$. In this case, for any $x \in \mathfrak{g}$, it is expressed as $x = x_S + x_R$ where $x_S \in S$ and $x_R \in R$, and the Lie bracket between any $x\,,y \in \mathfrak{g}$ is given as

$$[x\,,y]_\mathfrak{g} = [x_S\,,y_S] + [x_R\,,y_R] + \sigma(x_S)y_R - \sigma(y_S)x_R\,. \tag{1.4.17}$$

Note that if the Lie bracket consists only of the first two terms, then $\mathfrak{g} = S \oplus R$. Another remark is that $\sigma$ is a derivation for $[\cdot\,,\cdot]_S$, that is, $\sigma(x_S)$ is acting on the Lie bracket of radicals as

$$\sigma(x_S)[y\,,z]_R = [\sigma(x_S)\,y\,,z]_R + [y\,,\sigma(x_S)\,z]_R\,. \tag{1.4.18}$$

A crucial example of the Levi-Malcev decomposition is the Poincaré group $ISO(1,d-1)$. The Poincaré group is given by

$$ISO(1,d-1) = SO(1,d-1) \ltimes \mathbb{R}^{1\,,d-1} \tag{1.4.19}$$

where $SO(1,d-1)$ is called the Lorentz group, $\mathbb{R}^{1\,,d-1}$ is called the translation, and $\ltimes$ is called a semi-direct product. When we denote an element of $ISO(1,d-1)$ as $(\Lambda\,,a)$,

[1] The definition of centre $X$ is that $[X,Y]=0$ for all $Y \in \mathfrak{g}$.

where $\Lambda$ corresponds to the Lorentz and $a$ corresponds to the translation element, the the multiplication rule between two group elements is given by

$$(\Lambda\,, a) \cdot (\Lambda'\,, b) = (\Lambda \cdot \Lambda'\,, \Lambda\, b + a)\,. \tag{1.4.20}$$

This multiplication rule implies that $\mathfrak{so}(1, d-1)$ is the solvable part and $\mathbb{R}$ is corresponding to the radical part of the Poincaré algebra $\mathfrak{iso}(1, d-1)$. From (1.4.17), we can derive the Lie bracket relation for $\mathfrak{iso}(1, d-1)$

$$\begin{aligned} [x_S\,, y_S] &\;\to\; [M_{ab}\,, M_{cd}] = i(\eta_{bc} M_{ab} + \ldots)\,, \\ [x_R\,, y_R] &\;\to\; [P_a\,, P_b] = 0\,, \\ \sigma(x_S) y_R &\;\to\; [M_{ab}\,, P_c] = i(\eta_{cb}\, P_a - \eta_{ca}\, P_b)\,. \end{aligned} \tag{1.4.21}$$

From the Lie brackets, the adjoint representation of $M_{ab}$ is given as

$$\mathrm{ad}_{M_{ab}} = \left(\begin{array}{c:c} M_{..} & 0 \\ \hdashline 0 & \sigma_{\text{vect}} \end{array}\right)\,, \tag{1.4.22}$$

where $M$ block is a $\frac{1}{2}d(d-1)$-dimensional square matrix and $\sigma_{\text{vect}}$ is a $d$-dim. square matrix. On the other hand, the adjoint representation of $P_c$ is given as

$$\mathrm{ad}_{P_c} = \left(\begin{array}{c:c} 0 & 0 \\ \hdashline \sigma' & 0 \end{array}\right)\,. \tag{1.4.23}$$

This implies that adjoint representation is a trivial representation on $M$.

Now, we will introduce an important quantity for the semisimple groups: **Killing form**.

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**Definition | Killing form of Lie algrbra**

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A Killing form $\kappa$ is a map between direct product of identical Lie algebras and complex number

$$\kappa\,:\,\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}\,, \tag{1.4.24}$$

such that,

$$\kappa(x\,, y) = \mathrm{Tr}(\mathrm{ad}_X \cdot \mathrm{ad}_Y)\,, \tag{1.4.25}$$

where $\cdot$ denotes a matrix multiplication.

One property of the Killing form is that it is an $\mathrm{ad}$-invariant quantity, namely,

$$\kappa([x\,,y]\,,z) = \kappa(x\,,[y\,,z])\,. \tag{1.4.26}$$

From the definition of the Killing form, one can identify the Killing form of two generators are given as a multiplication of structure constants so that it is a symmetric matrix:

$$\kappa(T_a\,,T_b) = \kappa_{ab} = f_{ac}{}^{d}\,f_{bd}{}^{c}\,. \tag{1.4.27}$$

We will look at two Cartan's criterion related to the Killing form. The first one is the Cartan criterion for solvability and its statement is following: $\mathfrak{g}$ is solvable if and only if $\kappa(\mathfrak{g}\,,[\mathfrak{g}\,,\mathfrak{g}]) = 0$, or equivalently $\kappa(x\,,y) = 0$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}^{(1)}$.
Another criterion is the Cartan's criterion for semisimplicity: $\mathfrak{g}$ is semisimple if and only if $\kappa$ is non-degenerate, that is, $\kappa^{-1}$ is exists. One equivalent statement is that if $\kappa(x\,,y) = 0$ for fixed $x$ and for all $y \in \mathfrak{g}$, and if $\mathfrak{g}$ is a semisimple Lie algebra, then $x = 0$.

### 1.4.2 The Three Examples: $\mathfrak{su}(2)$, $\mathfrak{su}(3)$ and $\mathfrak{so}(N)$

Very first example of the representation theory of simple Lie algebra is the angular momentum problem in the quantum mechanics. Angular momentum operators can be considered as $\mathfrak{so}(3)$ operators [2] spanned by three generators $\{J_x\,,\ J_y\,,\ J_z\}$ which fulfill commutator relation $[J_i\,,\ J_j] = \varepsilon_{ijk}J_k$ where $\varepsilon_{ijk}$ is the Levi-Civita symbol. Since they are not commuting each other, so we choose a special direction among them (e.g. $J_z$) and combine two remaining generators linearly into ladder operator (e.g. $J_\pm = J_x \pm iJ_y$).
The main strategy is this: We consider one-dimensional eigenspace with respect to the chosen operator $J_z$ where the eigenstates are labeled by two quantum numbers $|j\,,\ m\rangle$. Then we define the highest state $|j, j\rangle$ as $J_+\,|j, j\rangle = 0$ and we find all other $2j + 1$ states by acting $J_-$ operator iteratively.
In summary, we could investigate the angular momentum states (or $\mathfrak{so}(3)$ representations) from (1) choosing reference direction and operators (called Cartan subalgebra), (2) finding suitable linear combination between remaining operators for ladder operator, (3) finding the state vanishing by raising operator $J_+$ (called highest-weight representation) and (4) acting lowering operator $J_-$ repeatedly.

Second example is —even though it is quite general— $\mathfrak{so}(d)$ algebra case.

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} - \delta_{ad}M_{bc} - \delta_{bc}M_{ad} + \delta_{bd}M_{ac})\,, \tag{1.4.28}$$

Similar to the $\mathfrak{so}(2)$ case, we consider some generators which will be referred as *Cartan generators*,

$$H_I = M_{2I-1\,2I}\,, \tag{1.4.29}$$

[2] Here, I will discuss about orbital angular momentum states. In the algebra level, $\mathfrak{su}(2)$ is identical to the $\mathfrak{so}(3)$ but in the group level, they are different. That is why $\mathfrak{su}(2)$ algebra admits half-integer representations.

where $I = 1, \cdots, r$ for $r = [d/2]$ called the rank of the group. Also, we define another two kinds of generators $E^{\pm}_{IJ}$ and $F^{\pm}_{IJ}$

$$\begin{aligned} E^{\pm}_{IJ} &= -\frac{i}{2}\big(M_{2I-1\,2J-1} \mp iM_{2I-1\,2J} - iM_{2I\,2J-1} \mp M_{2I\,2J}\big)\,, \\ F^{\pm}_{IJ} &= \frac{i}{2}\big(M_{2I-1\,2J-1} + \pm iM_{2I-1\,2J} + iM_{2I\,2J-1} - \mp M_{2I\,2J}\big)\,. \end{aligned} \tag{1.4.30}$$

When $d$ is odd, then there are $r$ more $E$ and $F$ generators:

$$\begin{aligned} E_K &= M_{2K-1\,d} - iM_{2K\,d}\,, \\ F_K &= M_{2K-1\,d} + iM_{2K\,d}\,. \end{aligned} \tag{1.4.31}$$

In this construction, we can rephrase Lie brackets (1.4.28) in terms of $H$, $E$, and $F$

$$\begin{gathered} [H_I\,, E^{\pm}_{JK}] = (\delta_{IJ} \pm \delta_{IK})E^{\pm}_{JK}\,, \quad [H_I\,, E_K] = \delta_{IK}E_K\,, \\ [H_I\,, J^{\pm}_{JK}] = -(\delta_{IJ} \pm \delta_{IK})F^{\pm}_{JK}\,, \quad [H_I\,, J_K] = -\delta_{IK}F_K\,, \\ [H_I\,, H_J] = 0\,. \end{gathered} \tag{1.4.32}$$

As following nomenclature in the $\mathfrak{so}(3)$ case, I will call $E$ and $F$ as raising and lowering operators, respectively. When you observe the Lie bracket between $H$ and, $E$ (or $F$), then you can find that result is proportional to $E$ (or $F$), that indicates that $E$ (or $F$) are eigenvectors of $H$, with some corresponding eigenvalues. For each generator, we can assign a $r$-component vector $\alpha$ whose components are given as eigenvalues of generator, and these vectors are called *root* vectors.
Finally, I will rephrase and develope former discussions with another Lie algebra $\mathfrak{su}(3)$. $\mathfrak{su}(3)$ algebra is spanned by Gell-Mann matrices.

$$\begin{gathered} \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{gathered} \tag{1.4.33}$$

Gell-Mann matrices are Hermitian matrices satisfying

$$\lambda_i\,\lambda_j = \frac{2}{3}\delta_{ij}\mathbb{1} + \sum_{k=1}^{8}(d_{ijk} + if_{ijk})\lambda_k\,, \tag{1.4.34}$$

where $d_{ijk}$ is symmetric tensor and their non-vanishing components are

$$\begin{gathered} d_{118} = d_{228} = d_{338} = -d_{888} = -2d_{448} = -2d_{558} = -2d_{668} = -2d_{778} = \frac{1}{\sqrt{3}} \\ d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = -d_{247} = -d_{366} = -d_{377} = \frac{1}{2} \end{gathered} \tag{1.4.35}$$

and $f_{ijk}$ is anti-symmetric tensor and their non-vanishing components are

$$f_{147} = f_{246} = f_{345} = f_{257} = -f_{156} = -f_{367} = \frac{1}{2}f_{123} = \frac{1}{\sqrt{3}}f_{458} = \frac{1}{\sqrt{3}}f_{678} = \frac{1}{2} \quad (1.4.36)$$

Let $T_i \equiv \frac{1}{2}\lambda_i$ be three-dimensional representation of $\mathfrak{su}(3)$ generators then they satisfy commutation and anti-commutation relations

$$[T_i\,, T_j] = i\sum_{k=1}^{8} f_{ijk}T_k\,, \quad \{T_i\,, T_j\} = \frac{1}{3}\delta_{ij}\mathbb{1} + \sum_{k=1}^{8} d_{ijk}T_k\,. \quad (1.4.37)$$

Now we will classify these generators as we did in the previous case. First, we find two Cartan generators

$$H_1 \equiv T_3 = \frac{1}{2}\lambda_3\,, \quad H_2 = T_8 = \frac{1}{2}\lambda_8\,, \quad (1.4.38)$$

then three raising operators $E_{i=1,2,3}$ and lowering operators $F_{i=1,2,3}$ are given as

$$\begin{aligned} E_1 &= T_1 + iT_2\,, \quad E_2 = T_4 + iT_5\,, \quad E_3 = T_6 + iT_7\,, \\ F_1 &= T_1 - iT_2\,, \quad E_2 = T_4 - iT_5\,, \quad E_3 = T_6 - iT_7\,. \end{aligned} \quad (1.4.39)$$

From the construction (1.4.38) and (1.4.39), we can rewrite $\mathfrak{su}(3)$ algebra in (1.4.37) as

$$\begin{aligned} &[H_1, E_1] = E_1\,, \quad [H_1\,, E_2] = \frac{1}{2}E_2\,, \quad [H_1\,, E_3] = -\frac{1}{2}E_3\,, \\ &[H_2\,, E_1] = 0\,, \quad [H_2\,, E_2] = \frac{\sqrt{3}}{2}E_2\,, \quad [H_2\,, E_3] = \frac{\sqrt{3}}{2}E_3 \end{aligned} \quad (1.4.40)$$

$$[E_1, F_1] = 2H_1\,,\ [E_2, F_2] = 2\left(\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2\right)\,,\ [E_3, F_3] = 2\left(-\frac{1}{2}H_1 + \frac{\sqrt{3}}{2}H_2\right)\,, \quad (1.4.41)$$

and

$$[E_1, E_3] = E_2\,, \quad [F_1, E_2] = E_3\,, \quad [F_3, E_2] = -E_1 \quad (1.4.42)$$

They are all non-vanishing commutation relations except some cases with $F_i$ but one can find easily by taking Hermitian conjugation with $E_i^\dagger = F_i$. Note that $H_1$ and $H_2$ are commuting each other, that is, they are forming abelian subalgebra.
When one looks (1.4.40) carefully, one can find that $E_j$ (or $F_j$) is an eigenvector of $[H_a, \cdot]$ operator with an eigenvalue. Let $\alpha$ is a two-component vector whose values are determined as the eigenvalue, then each generator is associated to a distinct $\alpha$, namely

$$E_1 \leftrightarrow \alpha_1 = (1, 0)\,, \quad E_2 \leftrightarrow \alpha_2 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\,, \quad E_3 \leftrightarrow \alpha_3 = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\,, \quad (1.4.43)$$

and $F_i$ are associated to $-\alpha_i$. We call $\alpha_i$ as a *root vector*. One can observe that commutator relation between $E$ and $F$ (1.4.41), the result is a linear combination between Cartan generators and their components are equivalent to root vectors'.
Another observation is that we can guess the results of commutator relation from the root vector. For instance, $[E_1, E_3] = E_2$ case, one can find that $\alpha_1 + \alpha_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) = \alpha_2$. On the

other hands, for $[E_1, E_2]$, $\alpha_1 + \alpha_2 = (\frac{3}{2}, \frac{\sqrt{3}}{2})$ and it does not associate any roots of $\mathfrak{su}(3)$ generators so it is a vanishing commutator. One can check this fact to other commutator relations like (1.4.42).
The final remark of this section is that since the root vectors of $\mathfrak{su}(2)$ are two-components vectors, so we can draw the root system of $\mathfrak{su}(2)$ on the paper as in 1.1:

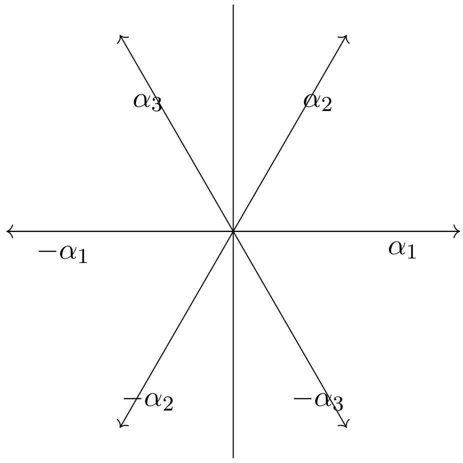


**Fig. 1.1**. Root system of $\mathfrak{su}(3)$

and this is related to the octet diagram of mesons or baryons because $SU(3)$ is the gauge group of strong interaction and mesons or baryons are representation of $SU(3)$. [3] That is how Gell-Mann received Nobel Prize in Physics in 1969.

### 1.4.3 Cartan Subalgebra and Root System

In the following discussions, we will focus on the simple Lie algebras and their representations. We observed many examples in the previous subsection 1.4.2 and the first step was finding a maximal set of abelian subalgebra, for example $H_1$ and $H_2$ generators in the $\mathfrak{su}(3)$ example. These generators are forming a subalgebra called a Cartan subalgebra

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**Definition | Cartan subalgebra**

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$\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of $\mathfrak{g}$ when
1. the maximal abelian subalgebra of $\mathfrak{g}$ and
2. any element of $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$, that is, there exists a basis of $\mathfrak{g}$, such that $\mathrm{ad}_{\mathfrak{h}}$ is diagonal for $\forall h \in \mathfrak{h}$.

In any simple Lie algebra, there exists a Cartan subalgebra. If $\mathfrak{h} = \mathrm{Span}\{H_i\}$ for $i = 1, \ldots, R$, than $R$ is called a rank of $\mathfrak{g}$, namely, $\dim \mathfrak{h} = \mathrm{rank}\ \mathfrak{g}$. For instance, $\mathfrak{so}(3)$, there is only one Cartan generator $L_0$ so its rank is $1$. Another example is $\mathfrak{so}(1,3)$ and we can choose two generators as Cartan like $\mathfrak{h}_{\mathfrak{so}(1,3)} = \mathrm{Span}\{M_{01}, M_{23}\}$ so its rank is $2$. One

[3] More precisely, mesons are Goldstone bosons of the spontaneous symmetry breaking of flavour symmetry $SU(3)_L \times SU(3)_R \to SU(3)$

remark that only for complex Lie algebra, all Cartan subalgebras are equivalent, that is, they can be conjugated by an automorphism of $\mathfrak{g}$.
Let $H_i \in \mathfrak{h}$ are Cartan generators and $E_\alpha \in \mathfrak{g}/\mathfrak{h}$ are remnant generators, or ladder operators in the previous subsection, then their Lie bracket is given as

$$[H_i\,, E_\alpha] = \alpha_i\, E_\alpha\,, \tag{1.4.44}$$

where $\alpha_i$ is called **roots** of $\mathfrak{g}$. Let us look a Lie bracket between two ladder operators $[E_\alpha\,, E_\beta]$. Using the Jacobi identity of Lie bracket, one can find

$$\begin{aligned}[H_i\,, [E_\alpha\,, E_\beta]] &= [[H_i\,, E_\alpha]\,, E_\beta] + [E_\alpha\,, [H_i\,, E_\beta]] \\ &= (\alpha_i + \beta_i)[E_\alpha\,, E_\beta]\,.\end{aligned} \tag{1.4.45}$$

Here are three different possibilities: If $\alpha_i + \beta_i = 0$ then $[E_\alpha\,, E_\beta]$ belongs to the Cartan subalgebra, while if $\alpha_i + \beta_i \neq 0$ then $[E_\alpha\,, E_\beta] \propto E_{\alpha+\beta}$. Finally, $\alpha_i + \beta_i$ is not a root, then $[E_\alpha\,, E_\beta] = 0$.
From the roots, we can construct the **root space** $\mathfrak{g}_\alpha$

$$\mathfrak{g}_\alpha \equiv \{X \in \mathfrak{g}\,|\, \mathrm{ad}_h X \equiv [h\,, X] = \alpha(h)\, X\,, \forall h \in \mathfrak{h}\}\ . \tag{1.4.46}$$

Note that $\alpha(h)$ is a number depending on the $h$, so $\alpha$ is an element of dual space of Cartan subalgebra. Using the root space, we can decompose the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\,, \tag{1.4.47}$$

for a set of all roots $\Phi$. That is, an element $x \in \mathfrak{g}$ will be decomposed as

$$x = x_0 + x_{\alpha^{(1)}} + x_{\alpha^{(2)}} + \cdots\,, \tag{1.4.48}$$

where $x_0 \in \mathfrak{h}$ is an Cartan subalgebra element and

$$x_\alpha = \sum_{k=1}^{\dim\,\mathfrak{g}_\alpha} c_k {T_k}^\alpha \tag{1.4.49}$$

for basis of $\mathfrak{g}_\alpha$, ${T_k}^\alpha$. When we restrict the Killing form into the Cartan subalgebra, we can find an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$, $i_\kappa$, explicitly,

$$\begin{aligned} i_\kappa\ : \mathfrak{h}\ &\to\ \mathfrak{h}^* \\ h &\mapsto \kappa(h\,, \cdot)\,, \end{aligned} \tag{1.4.50}$$

so $\kappa$ plays a similar role with metric tensor $g_{ab}$ as in the general relativity.
In the next few paragraph, we will show that $i_\kappa$ is really an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. From the Cartan criterion, $\kappa$ is a non-degenerated quantity but $\mathfrak{h}$ is just a subalgebra so it does not ensure about non-degeneracy of $i_\kappa$. Fortunately, we can prove the non-degeneracy of $i_\kappa$ from the following properties:
First of all, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to $\kappa$ (i.e. $\kappa(E_\alpha\,, E_\beta) = 0$) if $\alpha + \beta \neq 0$.

**Proof**

$$\kappa([h\,,E_\alpha]\,,E_\beta)+\kappa(E_\alpha\,,[h\,,E_\beta])=(\alpha(h)+\beta(h))\kappa(E_\alpha\,,E_\beta)\\ =(\alpha_i+\beta_i)h^i\kappa(E_\alpha\,,E_\beta)=0 \tag{1.4.51}$$

Ad-invariant property was used in this proof. ■
Moreover, $\mathfrak{g}_\alpha$ is orthogonal to the Cartan subalgebra $\mathfrak{h}$ for any non-zero root $\alpha$. It can be shown with the same argument to the above proof by replacing $E_\alpha$ to $h' \in \mathfrak{h}$.
Third, $\kappa_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate.

**Proof** Since Killing form is non-degenerate, there exists $x \in \mathfrak{g}$ such that $\kappa(h\,,x) \neq 0$. Let us writing $x$ as

$$x = x_{\mathfrak{h}} + \sum_\alpha x_\alpha\,, \tag{1.4.52}$$

with $x_{\mathfrak{h}} \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$, and using the previous propertiy of Killing form, we deduce that

$$\kappa(h\,,x) = \kappa(h\,,x_{\mathfrak{h}}) \neq 0\,. \tag{1.4.53}$$

This implies that there always exists $h' \in \mathfrak{h}$ which makes $\kappa(h\,,h') \neq 0$, and hence $\kappa_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate and (1.4.50) is really an isomorphism. ■
In a similar way, $\kappa_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-a}}$ is non-degenerate. One can repeat the above proof starting from existence of $y \in \mathfrak{h}$ such that $\kappa(E_\alpha\,,y) \neq 0$.
In the Cartan-Weyl basis, the Killing form will be given from the above properties

$$\kappa = \begin{pmatrix} \kappa_{ij} & 0 \\ 0 & \kappa_\alpha\,\delta_{\alpha+\beta\,,0} \end{pmatrix}\,, \tag{1.4.54}$$

where $\kappa_\alpha$ is a constant depending on the root $\alpha$ and $\kappa_{ij}$ is the Killing form restricted to $\mathfrak{h}$. In this construction, the isomorphism (1.4.50) will be given

$$i_\kappa(h) = \kappa_{ij}\,h^i\,\bar{H}^j\,,\quad i_\kappa^{-1}(\phi) = \kappa^{ij}\,\phi_i\,H_j \equiv H_\phi\,, \tag{1.4.55}$$

where $\bar{H}$ is the dual basis of $\mathfrak{h}$, that is, $\mathfrak{h}^* = \mathrm{Span}\left\{\bar{H}^i\right\}$, and $\phi = \phi_i\,\bar{H}^i \in \mathfrak{h}^*$. Using this isomorphism, one can define an inner product between two dual vectors of $\mathfrak{h}^*$ like

$$(\phi\,,\rho) = \kappa(i_\kappa^{-1}(\phi)\,,i_\kappa^{-1}(\rho)) = \kappa^{ij}\,\phi_i\,\rho_j\,, \tag{1.4.56}$$

and this makes us rewrite the commutator relation between $H_\alpha$ and $E_\beta$ such that

$$[H_\alpha\,,E_\beta] = \kappa^{ij}\,\alpha_i[H_i\,,E_\beta] = \kappa^{ij}\,\alpha_i\,\beta_j\,E_\beta = (\beta\,,\alpha)\,E_\beta\,. \tag{1.4.57}$$

With this inner product on the root space, we can also find the Lie bracket between two ladder operators with opposite roots

$$[E_\alpha\,,E_{-a}] = \kappa(E_\alpha\,,E_{-\alpha})\,H_\alpha\,, \tag{1.4.58}$$

where

$$H_\alpha \equiv i_\kappa^{-1}(\alpha) \equiv \kappa^{ij}\,\alpha_i\,H_j \in \mathfrak{g}\,. \tag{1.4.59}$$

Moreover, $\alpha$ has a non-vanishing norm, i.e. $(\alpha\,,\alpha) \neq 0$. These statements can be proven by non-degeneracy of the Killing form and Engel's theorem. But, here, we skip the proof of them.
Let us look the properties of the root system without proof.

First, root space $\mathfrak{g}_\alpha$ is a one-dimensional Lie algebra.
Second, if $\alpha \in \Phi$ then there is no $n\alpha \in \Phi$ except $n = \pm 1$. From this property, we can disjoint $\Phi$ into two pieces, $\Phi = \Phi_+ \cup \Phi_-$, where $\Phi_+$ is a set of roots such that their first non-vanishing coefficient in a basis is positive and *vice versa*. Also, we can classify the generators into three categories: The first category is $H_i$ which spans Cartan subalgebra. Other two types of generator exists for each $\alpha$, namely, $E_\alpha$ and $E_{-\alpha}$, usually called "raising" and "lowing" operators. This basis is called a **Cartan-Weyl basis**.
Third, there is a positive-definite scalar product on $\Phi$, that is, $(\alpha\,,\alpha) > 0$ for $\alpha \neq 0$.
Forth, the root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, spans $\mathfrak{h}^*$ the dual of Cartan subalgebra, namely, $\mathrm{Span}_{\mathbb{C}}(\Phi) = \mathfrak{h}^*$. In particular, this implies that $\mathrm{card}\,\Phi = \dim\,\mathfrak{h}^* = \dim\mathfrak{h}$. [4]
Fifth, the scalar product between two roots is always quotient number, $(\alpha\,,\beta) \in \mathbb{Q}$.
Sixth, If $(\beta\,,\alpha) < 0$ then $\alpha + \beta \in \Phi$. On the other hand, $\beta - \alpha \in \Phi$ for $(\beta\,,\alpha) > 0$.
Seventh, $\Phi$ is spanned by simple roots $\Phi_s$ which is a set of positive roots that cannot be written as the sum of two positive roots. From the fifth property, any root can be written as

$$\alpha = \sum_{i=1}^{r} c_i\,\alpha^{(i)}\,, \tag{1.4.60}$$

where $\alpha^{(i)}$ are simple roots and $c_i \in \mathbb{Q}$ are coefficients.

From the Seventh property of the simple root, we can define the **height** of a root:

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**Definition** | **Height of a root**

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The height ht$(\alpha)$ of a root $\alpha \in \Phi$ is given as a sum of its coefficient in the basis of simple roots $\Phi_s$. If $\alpha$ is given as $\alpha = \sum_{i=1}^{r} c_i\,\alpha^{(i)}$ then height is defined as

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{r} c_i\,. \tag{1.4.61}$$

Among the positive roots, there is one particular root $\Theta \in \Phi_+$, called the highest root of $\mathfrak{g}$, such that ht$(\Theta) >$ ht$(\alpha)$ for all other positive roots $\alpha$. As we will see later, the height of the highest root gives a characteristic of the algebra. Thus, we define the Coxeter number $\mathrm{g}$ of $\mathfrak{g}$ as

$$\mathrm{g} \equiv \mathrm{ht}(\Theta) + 1\,. \tag{1.4.62}$$

[4] Since $\Phi$ is a discrete set, the more natural notion for $\Phi$ is the cardinal which represents the number of elements of $\Phi$.

### 1.4.4 Root String and Its Characteristics

In the last section, we have discussed about root space. To describe root space, Cartan-Weyl basis is a good choice of the basis for Lie algebra. In this section, we continue the discussion on the root space to root string and classify the root string which is a stepstone for simple group classification.

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**Definition** | **Root string**

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Let $\alpha\,,\beta\in\Phi$ be two roots. The root string $S_{\beta\,,\alpha}$ is a set of roots such that

$$S_{\beta\,,\alpha}=\{\beta-n_-\,\alpha\,,\beta-(n_--1)\alpha\,,\cdots\,,\beta-\alpha\,,\beta\,,\beta+\alpha\,,\cdots\beta+(n_+-1)\alpha\,,\beta+n_+\,\alpha\}\,, \tag{1.4.63}$$

for $n_-\,,n_+\in\mathbb{N}$

The root string satisfies three properties.
*First*, there is no gap, that is, there is no such situation like $\{\,,\cdots\beta-2\alpha\,,\beta\,,\cdots\}$.
*Second*, if we consider *coroot*, given as

$$\alpha^\vee\equiv\frac{2}{(\alpha\,,\alpha)}\alpha\,, \tag{1.4.64}$$

then $(\beta\,,\alpha^\vee)=n_--n_+$.
*Third*, there are *at most* four elements in any root string. The last property can be proven by contradiction.

**Proof** Obviously, $S_{\pm\alpha\,,\alpha}$ has no more than four elements, so we exclude these cases in this proof. Suppose that $S_{\beta\,,\alpha}$ has five elements so that $S_{\beta\,,\alpha}=\{\beta-2\alpha\,,\beta-\alpha\,,\beta\,,\beta+\alpha\,,\beta+2\alpha\}$ up to redefinition. This implies that $(\beta+2\alpha)+\beta=2\,(\beta+\alpha)$ and $(\beta+2\alpha)-\beta=2\alpha$ are not roots due to the Second property of the root system in the previous subsection. As consequence, the root string $S_{\beta+2\alpha\,,\beta}$ only contains one elements so that $(\beta+2\alpha\,,\beta^\vee)=0$. This argument can be applied in a same way to $S_{\beta-2\alpha\,,\beta}$ so that $(\beta-2\alpha\,,\beta^\vee)=0$. By comparing two results, one concludes that $(\beta\,,\beta^\vee)=0$ but this is ridiculous according to the Third property of the root system in the previous subsection. In conclusion, $S_{\pm\alpha\,,\alpha}$ has no more than four elements. ■
*Forth*, there exists a relation between inner products such that

$$\frac{(\alpha+\beta\,,\alpha+\beta)}{(\beta\,,\beta)}=\frac{n_-+1}{n_+}\quad\Leftrightarrow\quad[(\beta\,,\alpha^\vee)+1]\left[1-\frac{(\alpha\,,\alpha)}{(\beta\,,\beta)}n_+\right]=0\,. \tag{1.4.65}$$

except $n_+=1$. We will prove this property explicitly.

Since there can be at most four elements in the root string, we can classify all root strings. Before we start the classification, we can write a Lie bracket in the Cartan-Weyl basis as

$$[E_\alpha\,,E_\beta]=\mathcal{N}_{\alpha\,,\beta}\,E_{\alpha+\beta}\,. \tag{1.4.66}$$

From the antisymmetric property of Lie bracket,

$$\mathcal{N}_{\alpha\,,\beta}=-\mathcal{N}_{\beta\,,\alpha} \tag{1.4.67}$$

Also, from the Jacobi identity,

$$\begin{aligned}&[E_\alpha\,,[E_\beta\,,E_\gamma]]+[E_\beta\,,[E_\gamma\,,E_\alpha]]+[E_\gamma\,,[E_\alpha\,,E_\beta]]=0\\=&\mathcal{N}_{\beta\,,\gamma}[E_\alpha\,,E_{\beta+\gamma}]+\mathcal{N}_{\gamma\,,\alpha}[E_\beta\,,E_{\gamma+\alpha}]+\mathcal{N}_{\alpha\,,\beta}[E_\gamma\,,E_{\alpha+\beta}]\,.\end{aligned} \tag{1.4.68}$$

If $\alpha+\beta+\gamma=0$ then (1.4.68) become

$$\mathcal{N}_{\beta\,,\gamma}\kappa_\alpha\,H_\alpha+\mathcal{N}_{\alpha\,,\beta}\kappa_\gamma\,H_\gamma+\mathcal{N}_{\gamma\,,\alpha}\,\kappa_\beta\,H_\beta=0\,, \tag{1.4.69}$$

where we used (1.4.58) with $\kappa_\alpha\equiv\kappa(E_\alpha\,,E_{-\alpha})$. Moreover, using

$$H_\gamma=i_\kappa^{-1}(\gamma)=i_\kappa^{-1}(-\alpha-\beta)=-i_\kappa^{-1}(\alpha)-i_\kappa^{-1}(\beta)=-H_\alpha-H_\beta\,, \tag{1.4.70}$$

the Jacobi identity (1.4.69) become

$$(\mathcal{N}_{\beta\,,\gamma}\,\kappa_\alpha-\mathcal{N}_{\alpha\,,\beta}\,\kappa_\gamma)H_\alpha+(\mathcal{N}_{\gamma\,,\alpha}\,\kappa_\beta-\mathcal{N}_{\alpha\,,\beta}\,\kappa_\gamma)\,H_\beta=0\,, \tag{1.4.71}$$

which implies

$$\mathcal{N}_{\beta\,,\gamma}\,\kappa_\alpha=\mathcal{N}_{\gamma\,,\alpha}\,\kappa_\beta=\mathcal{N}_{\alpha\,,\beta}\,\kappa_\gamma\,, \tag{1.4.72}$$

for any root $\alpha\,,\beta\,,\gamma\in\Phi$ such that their sum is zero. Next, consider $\beta=-\alpha$ and $\gamma=\beta-k\alpha$ case for $k\in\mathbb{N}_+$, then the Jacobi identity (1.4.68) becomes

$$\left(\mathcal{N}_{\alpha\,,\beta-k\alpha}\mathcal{N}_{\alpha\,,\beta-(k+1)\,\alpha}+\kappa_\alpha(\beta-k\alpha\,,\alpha)+\mathcal{N}_{\beta-k\alpha\,,-\alpha}\mathcal{N}_{\alpha\,,\beta-(k+1)\alpha}\right)\,E_{\beta-k\alpha}=0\,, \tag{1.4.73}$$

where we used (1.4.57). By using (1.4.67), (1.4.73) becomes

$$\mathcal{N}_{-\alpha\,,\beta-k\,\alpha}\mathcal{N}_{\alpha\,,\beta-(k+1)\,\alpha}=\mathcal{N}_{-\alpha\,,\beta-(k-1)\,\alpha}\mathcal{N}_{\alpha\,,\beta-k\alpha}+\frac{1}{2}\kappa_\alpha(\alpha\,,\alpha)(\beta-k\alpha\,,\alpha^\vee)\,. \tag{1.4.74}$$

Summing the left hand side of the above equation for $k=0$ to $n_-$, we obtain

$$\sum_{k=0}^{n_-}\mathcal{N}_{-\alpha\,,\beta-k\alpha}\mathcal{N}_{\alpha\,,\beta-(k+1)\alpha}=\sum_{k=1}^{n_-}\mathcal{N}_{-\alpha\,,\beta-(k-1)\,\alpha}\mathcal{N}_{\alpha\,,\beta-k\alpha}\,, \tag{1.4.75}$$

since $\beta-(n_-+1)\,\alpha$ is not a root as by the definition of the root string. Summing from $k=0$ to $n_-$, the second term of the right hand side of (1.4.74) gives

$$\sum_{k=0}^{n_-}(\beta-k\alpha\,,\alpha^\vee)=(n_-+1)(\beta\,,\alpha^\vee)-n_-(n_-+1)=-n_+(n_-+1)\,, \tag{1.4.76}$$

with the root string property (1.4.64). Combining all results, we can find the the relation from the Jacobi identity

$$\mathcal{N}_{\alpha\,,\beta}\mathcal{N}_{-\alpha\,,\beta+\alpha}=\frac{1}{2}\kappa_\alpha\,(\alpha\,,\alpha)\,n_+(n_-+1)\,. \tag{1.4.77}$$

From the previous identity (1.4.72), one can show that $\mathcal{N}_{-\alpha\,,\alpha+\beta}\,\kappa_\beta=\mathcal{N}_{-\alpha\,,-\beta}\,\kappa_{\alpha+\beta}$ then the above relation is rewritten as

$$\mathcal{N}_{\alpha\,,\beta}\mathcal{N}_{-\alpha\,,-\beta}=-\frac{1}{2}n_+(n_-+1)(\alpha\,,\alpha)\frac{\kappa_\alpha\,\kappa_\beta}{\kappa_{\alpha+\beta}}\,. \tag{1.4.78}$$

Using the fourth property of root string (1.4.65), the above relation can be simplified

$$\mathcal{N}_{\alpha\,,\beta}\mathcal{N}_{-\alpha\,,-\beta} = -\frac{1}{2}(n_-+1)^2\frac{\kappa_\alpha\,\kappa_\beta}{\kappa_{\alpha+\beta}}\frac{(\alpha\,,\alpha)(\beta\,,\beta)}{(\alpha+\beta\,,\alpha+\beta)}\,. \tag{1.4.79}$$

This above relation implies a relationship between $\mathcal{N}_{\alpha\,,\beta}$ and $\mathcal{N}_{-\alpha\,,-\beta}$.
From the above properties, we can classify the root string. As the consequences of the second property (1.4.64) and third property, one can show

$$(\beta\,,\alpha^\vee) = n_- - n_+ \in \{0\,,\pm1\,,\pm2\,,\pm3\}\,. \tag{1.4.80}$$

Let us introduce the angle parameter $\theta_{\alpha\beta}$, defined from the inner product between two roots:

$$(\alpha\,,\beta) = \sqrt{(\alpha\,,\alpha)(\beta\,,\beta)}\cos\theta_{\alpha\beta}\,, \tag{1.4.81}$$

so we can rewrite

$$(\beta\,,\alpha^\vee)(\beta^\vee\,,\alpha) = 4\cos^2\theta_{\alpha\beta} = \{0\,,1\,,2\,,3\}\,. \tag{1.4.82}$$

Here, we ignore $\cos\theta_{\alpha\beta} = 1$ since it is too trivial, that is, $\alpha \propto \beta$ so $\beta = \pm\alpha$. Then all possible value of $((\beta\,,\alpha^\vee)\,,(\beta^\vee\,,\alpha))$ is $(1\,,2)\,,(-1\,,-2)\,,(1\,,3)\,,(-1\,,-3)$ and $(0\,,0)$ without regarding the order. For each pair, the angle $\theta_{\alpha\beta}$ can be determined from

$$(\beta\,,\alpha^\vee) = \frac{(\alpha\,,\alpha)}{(\beta\,,\beta)}\,(\beta^\vee\,,\alpha)\,. \tag{1.4.83}$$

While $(\beta\,,\alpha^\vee) = n_- - n_+$ and keeping in mind that $n_- + n_+ \le 3$, we can find all possible root strings as follow table

| $(\beta\,,\alpha^\vee)$ | $(\beta^\vee\,,\alpha)$ | $\theta_{\alpha\beta}$ | $\frac{(\alpha\,,\alpha)}{(\beta\,,\beta)}$ | $(n_-\,,n_+)$ |
|---|---|---|---|---|
| $0$ | $0$ | $\frac{\pi}{2}$ | $1$ | $(1\,,1)$ |
| $1$ | $1$ | $\frac{\pi}{3}$ | $1$ | $(2\,,1)$ or $(1\,,0)$ |
| $-1$ | $-1$ | $\frac{2\pi}{3}$ | $1$ | $(1\,,2)$ or $(0\,,1)$ |
| $2$ | $1$ | $\frac{\pi}{4}$ | $\frac{1}{2}$ | $(2\,,0)$ |
| $1$ | $2$ | $\frac{\pi}{4}$ | $2$ | $(2\,,1)$ or $(1\,,0)$ |
| $-2$ | $-1$ | $\frac{3\pi}{4}$ | $\frac{1}{2}$ | $(0\,,2)$ |
| $-1$ | $-2$ | $\frac{3\pi}{4}$ | $2$ | $(1\,,2)$ or $(0\,,1)$ |
| $3$ | $1$ | $\frac{\pi}{6}$ | $\frac{1}{3}$ | $(3\,,0)$ |
| $1$ | $3$ | $\frac{\pi}{6}$ | $3$ | $(2\,,1)$ or $(1\,,0)$ |
| -3 | -1 | $\frac{5\pi}{6}$ | $\frac{1}{3}$ | $(0\,,3)$ |
| -1 | -3 | $\frac{5\pi}{6}$ | 3 | $(1\,,2)$ or $(0\,,1)$ |

**Table 1.1**. All possible non-trivial root strings

One can easily show that the fourth property of the root string (1.4.65) is satisfied from the Table 1.1 except $n_+ = 0$ cases.

Now, go back to the structure constant $\mathcal{N}$. The Second property of the root system in the previous subsection tells us that the root space is invariant under $\alpha \to -\alpha$. Let us define an automorphism $\theta$ of $\mathfrak{g}$ which flipping all roots,

$$\theta([x\,,y]) = [\theta(x)\,,\theta(y)]\,, \quad \theta(\lambda\, x + y) = \lambda\,\theta(x) + \theta(y)\,, \quad \theta^2 = \mathbb{1}\,. \tag{1.4.84}$$

Let us write the action of $\theta$ to the generators

$$\theta(E_\alpha) = \varepsilon_\alpha\, E_{-a}\,, \quad \theta(H_\alpha) = \eta_\alpha\, H_\alpha\,, \tag{1.4.85}$$

where $\varepsilon_\alpha$ and $\eta_\alpha$ are numbers depending on the root $\alpha$. From the third condition of $\theta$ in (1.4.84), one can find

$$\varepsilon_\alpha\,\varepsilon_{-\alpha} = 1\,, \quad \eta_\alpha^2 = 1\,. \tag{1.4.86}$$

Acting $\theta$ to the Lie bracket between a ladder operator and Cartan generaotr $[H_\alpha\,, E_\beta] = (\beta\,,\alpha)\,E_\beta$, then

$$[H_\alpha\,, E_{-\beta}] = \frac{1}{\eta_\alpha}(\beta\,,\alpha)E_{-\beta}\,. \tag{1.4.87}$$

Since we require that $\theta$ should preserve the Lie bracket, we can find $\eta_\alpha = -1$. Applying $\theta$ to the Lie bracket of two ladder operators $E_\alpha$ and $E_\beta$ such that $\alpha + \beta \neq 0$,

$$[E_{-\alpha}\,, E_{-\beta}] = \frac{\varepsilon_{\alpha+\beta}}{\varepsilon_\alpha\,\varepsilon_\beta}\mathcal{N}_{\alpha\,,\beta}\,E_{-\alpha-\beta}\,. \tag{1.4.88}$$

Again the preservation of the Lie bracket tells us the relation

$$\mathcal{N}_{-\alpha\,,-\beta} = \frac{\varepsilon_{\alpha+\beta}}{\varepsilon_\alpha\,\varepsilon_\beta}\mathcal{N}_{\alpha\,,\beta}\,. \tag{1.4.89}$$

Even though there is no further relation between $\mathcal{N}_{-\alpha\,,-\beta}$ and $\mathcal{N}_{\alpha\,,\beta}$, we can fix the structure constant by imposing all $\varepsilon_\alpha = 1$ or $\varepsilon_\alpha = -1$, that is, $\mathcal{N}_{\alpha\,,\beta} = \pm\mathcal{N}_{-\alpha\,,-\beta}$. If we choose the second option and applying it to (1.4.79), then we obtain

$$\mathcal{N}_{\alpha\,,\beta}^2 = \frac{1}{2}(n_- + 1)^2\frac{\kappa_\alpha\,\kappa_\beta}{\kappa_{\alpha+\beta}}\frac{(\alpha\,,\alpha)(\beta\,,\beta)}{(\alpha+\beta\,,\alpha+\beta)}\,. \tag{1.4.90}$$

Further simplification is possible by choosing appropriate basis by fixing $\kappa_\alpha$ for each root $\alpha$. One choice is called the **Chevalley basis** such that

$$\kappa_\alpha = \frac{2}{(\alpha\,,\alpha)}\,, \tag{1.4.91}$$

so the structure constant is given as

$$\mathcal{N}_{\alpha\,,\beta} = \pm(n_- + 1)\,, \tag{1.4.92}$$

where the sign is depending on the couple of roots $\alpha$ and $\beta$.

### 1.4.5 Dynkin Diagram

In this section, we will discuss about the Dynkin diagram and classification of the simple Lie algebras. To construct the Dynkin diagram, we should start from the Cartan matrix which contains all information about the simple Lie algebra, the we will work with Chevalley-Serre basis which is an adoption of Chevalley basis to the Cartan-Weyl basis.
First of all, the definition of **Cartan matrix** of $\mathfrak{g}$ is following:

---

**Definition** | **Cartan matrix**

---

A Cartan matrix of simple Lie algebra is a $r \times r$ matrix with integer entries and satisfying following properties:

- $A_{ii} = 2$, for $i = 1\,, \cdots, r$
- $A_{ij} \in \mathbb{Z}_{\leq 0}$, for $i\,, j = 1\,, \cdots, r$ with $i \neq j$
- If $A_{ij} = 0$, then $A_{ji} = 0$.
- $\det\, A > 0$
- $A$ is not block diagonal.

In the following discussions, following explicit form of the Cartan matrix will be used:

$$A^{ij} = (\alpha^{(i)\vee}\,, \alpha^{(j)}) = 2\frac{(\alpha^{(i)}\,, \alpha^{(j)})}{(\alpha^{(i)}\,, \alpha^{(i)})}\,. \tag{1.4.93}$$

For further convenience, let us denote $h_i \equiv H_{\alpha^{(i)\vee}}$, $e_i \equiv E_{\alpha^{(i)}}$, and $f_i \equiv E_{-\alpha^{(i)}}$ where $\{\alpha^{(i)}\}$ are the simple roots of the Lie algebra with $i = 1\,, \cdots, r$. In this setup, the Lie algebra between three generators is given as

$$[h_i\,, h_j] = 0\,, \quad [h_i\,, e_j] = A^{ij}\, e_j\,, \quad [h_i\,, f_j] = -A^{ij}\, f_j\,, \quad [e_i\,, f_j] = \delta_{ij}\, h_i\,. \tag{1.4.94}$$

There are remnant generators which are not associated with simple roots. Fortunately, remaining relation can be found from the **Serre relation**, given as

$$\mathrm{ad}_{e_i}^{1-A_{ij}}\, e_j = 0\,, \quad \mathrm{ad}_{f_i}^{1-A_{ij}}\, f_j = 0\,. \tag{1.4.95}$$

Through those relations (1.4.94) and (1.4.95), called Chevalley-Serre relation, the structure of root space is completely understood, that is, Cartan matrix contains enough information to understand the simple Lie algebra.
For the Kac-Moody algebra, an infinite-dimensional Lie algebra, relation between generators are given by the Serre relation and the generalised Cartan matrix, which satisfies all properties of the Cartan matrix except the determinant condition, that is, allowing $\det A \leq 0$ case.
The entire root system can be driven from the positive roots due to the following propositions:
<u>a</u>: Let $\mathcal{P} = \{\alpha_i\}$ be a set of $n$ positive roots such that $(\alpha_k\,, \alpha_l) \leq 0\,,\ \forall k \neq l$. Then the $n$ roots of $\mathcal{P}$ are linearly independent.

**Proof** Suppose that the set of positive roots $\mathcal{P}$ are linearly dependent, say,

$$\alpha_n = \gamma + \delta\,, \tag{1.4.96}$$

with

$$\gamma = \sum_{i=1}^{k-1} c_i \alpha_i\,, \quad \delta = \sum_{j=k}^{r-1} d_j \alpha_j\,, \tag{1.4.97}$$

where $c_i > 0$ and $d_j \leq 0$. Since $\alpha_n$ is a positive root, then $\gamma \neq 0$. Moreover, the scalar product between $\gamma$ and $\delta$ is always positive or zero

$$(\gamma\,,\delta) = \sum_{i=1}^{k-1}\sum_{j=k}^{r-1} c_i d_j (\alpha_i\,,\alpha_j) \geq 0\,, \tag{1.4.98}$$

from the assumption $(a_i\,,a_j) \leq 0$ for any $i \neq j$, so

$$(\alpha_r\,,\gamma) = (\gamma\,,\gamma) + (\gamma\,,\delta) \geq 0\,. \tag{1.4.99}$$

On the other hand, the direct calculation of the scalar product is given as

$$(\alpha_r\,,\gamma) = \sum_{i=1}^{k-1} c_i (\alpha_r\,,\alpha_i) \leq 0\,. \tag{1.4.100}$$

This consistency is originated from the assumption in the beginning. Therefore, the set $\mathcal{P} = \{\alpha_i\}$ is the set of linearly independent vectors. ■

b: If two simple roots are orthogonal, then their sum is not a root, namely, $(\alpha^{(i)}\,,\alpha^{(j)}) = 0$ then $\alpha^{(i)} + \alpha^{(j)} \notin \Phi_+$ for $i \neq j$.

**Proof** [[proof]]

c: Let $\beta \in \Phi_+/\Phi_s$ be a positive root which are not simple. Then there exists a simple roots $\alpha^{(k)} \in \Phi_s$ such that the difference $\beta - \alpha^{(k)}$ is still a positive root.

**Proof** [[proof]]

From the proposition c, an important corollary exists:
d: Let $\beta \in \Phi_+$ such that ht$(\beta) = n > 1$. Then there exists $\beta' \in \Phi_+$ with ht$(\beta') = n - 1$ and a simple root $\alpha^{(k)} \in \Phi_s$ such that $\beta = \beta' + \alpha^{(k)} \in S_{\beta'\,,\alpha^{(k)}}$.
e If $\Phi_+$ does not contain any root of ht $= m$ for some integer $m > 1$, then there exist no root of height greater than $m$ at all.

**Proof** [[proof]]

From a to e, it can be shown that the positive roots provide enough information to recover whole root system. First, the simple roots are the only roots with the unity height, by definition.
Second, roots of height two should have the form $\alpha^{(i)} + \alpha^{(j)}$ for $i \neq j$. From the Sixth property in Sec.1.4.3, if scalar product between two roots is negative, then $\alpha^{(i)} + \alpha^{(j)} \in \Phi$.

On the other hand, if scalar product between two roots is vanished, then the sum of two roots is not a root, from b. In any cases, the Cartan matrix element $A^{ij}$ (or $A^{ji}$) gives the criterion.

Third, let us denote a set of roots with height less or equal to $n$ as $\Phi_{+|n}$. From d, any root $\gamma \in \Phi_{+|n}$ is given as $\beta + \alpha^{(k)}$ for a given $\beta \in \Phi_{+|n-1}$ and a simple root $\alpha^{(k)}$. Therefore, to find all roots of $\Phi_{+|n}$, it is sufficient to compute all possible root strings $S_{\beta\,,\alpha^{(k)}}$ for $\beta \in \Phi_{+|n-1}$ and $\alpha^{(k)} \in \Phi_s$. The third procedure can be repeated before reach to the maximal height. Then all positive roots are obtained and negative roots are easily found as opposite of the positive roots.

The Cartan matrix gives not only the information for the root system of the group, but also useful tools for classifying the simple group, which we will see soon, and construction of the representation, which we will see in the next subsection, Sec. 1.4.6. In the next subsection, we will revisit this procedure.

A **Dynkin diagram** is a way to describe the Cartan matrix in a graphical way. A Dynkin diagram obeys the following rules:

- To each simple root correspond a node in the diagram.
- Only roots with no vanishing scalar product are connected.
- The number of line $\ell$ connecting two nodes corresponding to the roots $\alpha^{(i)}$ and $\alpha^{(j)}$ encodes the product of two Cartan matrix elements involving, through

$$\ell = \frac{4(\alpha^{(i)}\,,\alpha^{(i)})^2}{(\alpha^{(i)}\,,\alpha^{(j)})(\alpha^{(j)}\,,\alpha^{(j)})} = A^{ij}\,A^{ji}\,. \tag{1.4.101}$$

- If there are more than one line connecting two nodes, an arrow is drawn between them, from the longer to the shorter root.

For instance, $\mathfrak{su}(3)$ case, there are two simple roots $\alpha_1$ and $\alpha_3$, given in 1.1, and the Cartan matrix is given as

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\,, \tag{1.4.102}$$

and the corresponding diagram can be drawn as following:

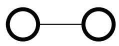
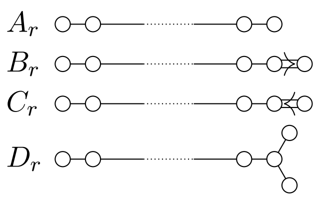


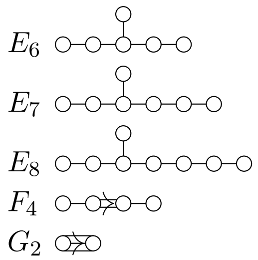
**Fig. 1.2**. Dynkin diagram for $A_2 = \mathfrak{su}(3)$.

All possible simple Lie algebras are classified by Dynkin diagram (or Cartan matrix). Four classical Lie algebra families, denoted $A_r$, $B_r$, $C_r$ and $D_r$ are parametrised by $r$, the rank of the Lie algebra.




(1.4.103)

which corresponding to $\mathfrak{su}(r+1)$, $\mathfrak{so}(2r+1)$, $\mathfrak{sp}(2r\,,\mathbb{R})$ and $\mathfrak{so}(2r)$, respectively [5]. Also, there are five exceptional Lie algebras, denoted as $E_6\,,\ E_7\,,\ E_8\,,\ F_4$, and $G_2$.




(1.4.104)

Note that one can easily find the following isomorphisms between Lie algebras from the Dynkin diagram:

- $A_1 \simeq B_1 \simeq C_1$
- $B_2 \simeq C_2$
- $D_2 \simeq A_1 \times A_1$
- $D_3 \simeq A_3$
- $E_3 \simeq A_1 \times A_2$
- $E_4 \simeq A_4$
- $E_5 \simeq D_5$

### 1.4.6 Weight and Highest Weight

One usual way to construct them is called the highest weight representation which is partially discussed in the Subsection 1.4.2. Through the following three subsections, we will discuss about the highest weight representation and their classification. Here, we will mainly introduce a concept of weights and highest weight and discuss about their properties.

Let us suppose to be $V$ is a $\mathfrak{g}$-module of representation $\rho$ which basis is chosen as $\rho(H_\alpha)$ acts diagonally, say,

$$\rho(H_\alpha)\, v^a = \lambda_i^a\, v^a\,, \tag{1.4.105}$$

where $v^a$ is a component of $v \in V$. Let $h = \sum_i c^i\, H_\alpha \in \mathfrak{h}$ and once $\rho(h)$ is acted on $v^a$ then one can find

$$\rho(h)\, v^a = \left(\sum_i c_i\, \lambda_i^a\right) \phi^a\,. \tag{1.4.106}$$

From this relation, a quantity $W \in \mathfrak{h}^*$ can be defined as

$$W^a\left(\sum_i c_i\, h_i\right) = \sum_i c_i\, \lambda_i^a\,, \tag{1.4.107}$$

[5]More precisely, this classification can be applied to the complex Lie algebra. For complex Lie algebra, $A_r \simeq \mathfrak{sl}_{\mathbb{C}}\, r+1\,,\ B_r \simeq \mathfrak{so}_{\mathbb{C}}\, 2r+1\,,\ C_r \simeq \mathfrak{sp}_{\mathbb{C}}\, 2r\,$, and $D_r \simeq \mathfrak{so}_{\mathbb{C}}\, 2r$.

which is called **weight vectors**. In other words, the weight is the eigenvalue of a vector in the module with respect to the Cartan subalgebra element. Recall the Subsection 1.4.3, the root vectors $\alpha$ are also dual vector of the Cartan subalgebra. In fact, the root vectors are a special case of weight vectors associated with the adjoint representation. Similarly to the root system, the **weight space** $V_\alpha \subset V$ can be constructed as

$$V_W \equiv \{v \in V \,|\, \rho(h)\, v = W(h)\, v\,, \forall h \in \mathfrak{h}\}\,, \tag{1.4.108}$$

and weight vectors decompose the vector module $V$, such that

$$V = \bigoplus_W V_W\,. \tag{1.4.109}$$

Again, $W$ can be regarded as a vector of $\mathfrak{h}^*$.
We observed the construction of $\mathfrak{su}(2)$ Lie algebra using the raising and lowing operators $J_+$ and $J_-$. It can be generalised by the weight. Let $v^a \in V$ is a $\mathfrak{g}$-module of representation $\rho$ and $W^a$ is a weight vector associate with the vector $v^a$. Then weight of new vector $\rho(E_\alpha)\, v^a$ is measured by acting $\rho(h)$ to it, that is,

$$\begin{aligned} \rho(h)\, E_\alpha\, v^a &= (\rho(E_\alpha)\,\rho(h) + \alpha(h)\,\rho(E_a))\, v^a \\ &= (W^a(h) + \alpha(h))\,\rho(E_a)\, v^a\,, \end{aligned} \tag{1.4.110}$$

which implies that $\rho(E_\alpha)\, v^a$ has the weight $W^a + \alpha$ if $\rho(E_\alpha)\, v^a \neq 0$. In this sense, $E_\alpha$ and $E_{-\alpha}$ are regarded as raising and lowering operators, respectively.
For root vectors, a root string (1.4.63) has been considered in the subsection 1.4.4 and here a weight string can be considered as

$$\{W^*\,, W^* - \alpha\,, \cdots\,, W^* - n\,\alpha\}\,, \tag{1.4.111}$$

Let $v_0$ is a vector with weight $W^*$ then $v_j \equiv (\rho(E_{-\alpha}))^j\, v_0$ should be a vector with weight $W^* - j\,\alpha$. When we demand that the weight string is finished for $W^* - n\,\alpha$, then

$$\rho(E_{-a})\, v_n \overset{!}{=} 0\,. \tag{1.4.112}$$

To find $n$, first let us consider a vector $\rho(E_\alpha)\, v_k = r_k\, v_{k-1}$ then one can find

$$\begin{aligned} \rho(E_\alpha)\, v_k &= r_k\, v_{k-1} \\ &= \rho(E_\alpha)\,\rho(E_{-a})\, v_{k-1} \\ &= (\rho(E_{-\alpha})\,\rho(E_\alpha) + \kappa_\alpha\,\rho(H_\alpha))\, v_{k-1} \\ &= r_{k-1}\, v_{k-1} + \kappa_\alpha\,[(W^*, \alpha) - (k-1)(\alpha\,, \alpha)]\, v_{\kappa-1} \\ &= [r_{k-1} + \kappa_\alpha\,((W^*, \alpha) - (k-1)(\alpha\,, \alpha))]\, v_{\kappa-1}\,. \end{aligned} \tag{1.4.113}$$

where we used (1.4.58) and the notation (1.4.56). Once we impose $r_0 = 0$ then we find

$$r_k = \kappa_\alpha\,(k\,(W^*, \alpha) - \frac{1}{2} k(k-1)(\alpha\,, \alpha))\,. \tag{1.4.114}$$

The condition (1.4.112) says that $\rho(E_\alpha)\,\rho(E_{-\alpha})\,v_n = r_{n+1}\,v_q = 0$, and, as consequences, the length of weight string $n$ is given as

$$n = \frac{2(W^*, \alpha)}{(\alpha\,,\alpha)}\,. \tag{1.4.115}$$

In the weight string, there exists a hidden symmetry, called **Weyl reflection** $S_\alpha$, given as

$$S_\alpha : \quad W \;\to\; W' = W - \frac{2(W, \alpha)}{(\alpha\,,\alpha)}\alpha\,, \tag{1.4.116}$$

and a set $S_\alpha$ forms a group, called the Weyl group $\mathcal{W}$. Geometrically speaking, the Weyl reflection $S_\alpha$ is a reflection in the weight space with respect to the hyperplane orthogonal to the $\alpha$. In the $\mathfrak{su}(3)$ root space Fig.1.3, for instance, the Weyl reflections are denoted as dashed lines. The dashed line on the $y$-axis is corresponding to the Weyl reflection $S_{\alpha_1}$, and so on. One comment of this geometrical interpretation is that an isolated region generated by the hyperplanes is called the Weyl chamber. For instance, in 1.3, six triangular regions divided by dashed lines correspond to the Weyl chamber.

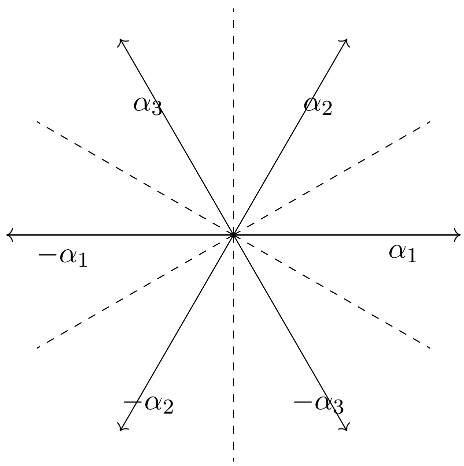



**Fig. 1.3**. Weyl reflection in root system of $\mathfrak{su}(3)$

Now we will discuss about the **highest weight representation**. The starting point is, of course, the highest weight vector.

**Definition | Highest weight vector**

Let $V$ suppose to a $\mathfrak{g}$-module and $\rho$ supposed to be a representation. A non-zero vector $v \in V$ is called a **highest weight vector** if it is an eigenvector of $\rho(h)$ and it is in the kernel of $\rho(X)$, where $X \in \mathfrak{g}_{\alpha\in\Phi^+}$.

In other words, highest weight vector is a weight vector which is vanished by all raising operators. For example, $\mathfrak{su}(2)$ case, the state $|j, j\rangle$ is the highest vector. There are three propositions about the representation of semisimple Lie algebra $\mathfrak{g}$ and its highest weight vector. We show these propositions without proof.

(i): Every finite-dimensional $\mathfrak{g}$-module $V$ possesses a highest weight vector.
(ii): The subspace $W \subset V$ generated by the images of a highest weight vector $v$ under successive applications of $\rho(X)$ where $X \in \mathfrak{g}_{\alpha\in\Phi^-}$ is a $\mathfrak{g}$-module of irreducible subrepresentation.
(iii): An irreducible representation possesses a unique highest weight vector up to scalars.

Proposition (i)-(iii) tell us a way to construct irreducible representation from the highest weight vector. From the Seventh property of the root system in the Sec.1.4.3, only the negative simple roots are enough to construct whole irreducible representation. So, any $\mathfrak{g}$-module $V$ of irreducible representation $\rho$ is generated by the images of its highest weight vector $v$ successive application of $\rho(X)$ where $X \in \mathfrak{g}_{\alpha\in-\Phi_s}$.
As we saw in (1.4.115), the norm between the weight $W$ and the root $\alpha$ is an important quantity. Also, it implies that the basis of the weight vectors can be given as the dual basis of the $\alpha^\vee$, namely,

$$W = \sum_{i=1}^{r} w^i \, W_{(i)} \, , \tag{1.4.117}$$

where $W_{(i)}$ is the basis of the weight vector, called the **fundamental weight**, satisfying

$$(W_{(i)} \, , \alpha^{(j)\vee}) = \delta_i^j \, , \tag{1.4.118}$$

and $W^i$ is the number which is called the **Dynkin coefficient** or **Dynkin label** and given as

$$w^i = 2\frac{(W \, , \alpha_i)}{(\alpha_i \, , \alpha_i)} \, , \tag{1.4.119}$$

Through the Dynkin label, all weights can be expressed as the linear combination of roots from the Cartan matrix, so all irreducible representations can be analysed by the Dynkin label. Luckily, the following theorem also can help to construct the highest weight representation.

**Theorem | Dynkin label of highest weight**

For every irreducible representation, the highest weight $W$ can be written as

$$\Lambda = \sum_{i=1}^{r} w^i \, W_{(i)} \, , \tag{1.4.120}$$

where $w^i \in \mathbb{Z}_{\geq 0}$. In addition, there exists an irreducible representation with highest weight given by (1.4.120).

We accept this theorem without any proof. By the proposition (iii), this representation is unique. This implies that one can construct irreducible representation by taking any Dynkin label with positive integer value.
Also, the dimension of representation is given by the Weyl's dimensionality fomula:

$$\dim \rho = \prod_{\alpha\in\Phi_+} \frac{(W + \Lambda, \alpha)}{(\Lambda \, , \alpha)} \, , \tag{1.4.121}$$

where vector $\Lambda$ is called the Weyl vector and given as

$$\Lambda = \sum_i W_{(i)} \,. \tag{1.4.122}$$

Let us look at the representations of $A_2 \simeq \mathfrak{su}(3)$ [6]. From the Cartan matrix (1.4.102), the fundamental weights are given as

$$W_{(1)} = \frac{2}{3}\alpha^{(1)} + \frac{1}{3}\alpha^{(2)} \,, \quad W_{(2)} = \frac{1}{3}\alpha^{(1)} + \frac{2}{3}\alpha^{(2)} \,. \tag{1.4.123}$$

Using the root system (1.4.43), the simple roots and the fundamental weights are given by

$$\begin{aligned} \alpha^{(1)} &= (1,0)\,, \quad \alpha^{(2)} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})\,, \\ W_{(1)} &= (\frac{1}{2}, \frac{\sqrt{3}}{6})\,, \quad W_{(2)} = (0, \frac{\sqrt{3}}{3})\,. \end{aligned} \tag{1.4.124}$$

One of the most simplest representation of $A_2$ is given by the highest weight:

$$\boxed{1 \quad 0}$$

Since the first component is positive, $\rho(E_{-\alpha_1})$ should be acted to the highest weight and we obtain the weight:

$$\boxed{-1 \quad 1}$$

Again, the second Dynkin label is positive, and the vector with lower weight can be obtained by acting $\rho(E_{-\alpha_2})$:

$$\boxed{0 \quad -1}$$

In this weight, there is no positive Dynkin label. This facts implies that this weight is the lowest weight, so we stop here. Finally three-dimensional representation has been found:

$$\boxed{1 \quad 0}$$

$$\boxed{-1 \quad 1}$$

$$\boxed{0 \quad -1}$$

We can check the Weyl's dimensionality formula (1.4.121) in this case. Actually, one can find a dimensionality formula for $\mathfrak{su}(3)$ representation.

---

**Exercise 2| Dimensionality formula for $\mathfrak{su}(3)$**

---

Show that the dimensionality formula for $\mathfrak{su}(3)$ can be given as

$$\dim \rho_{(w^1, w^2)} = \frac{1}{2}(w^1+1)(w^2+1)(w^1+w^2+2)\,, \tag{1.4.125}$$

[6] We changed the notation for the root as $\alpha^3 = \alpha^{(2)}$ for the convenience. In this notation, $\alpha^2 = \alpha^{(1)} + \alpha(2)$.

where $\rho_{(w^1,w^2)}$ is the irreducible representation with the highest weight $(w^1\,,w^2)$.

In physics, the highest weight representations are appeared in many places. The first example is the angular momentum states in the quantum mechanics. Another crucial application is the Verma module in the conformal field theory. Since the isometry of the conformal field theory, $SO(2,d)$ for $d$-dimensional CFT, is non-compact group, there is no finite dimensional unitary irreducible representation. Despite of absence of the finite-dimensional representation, a vector module of $\mathfrak{so}(2,d)$ can be constructed in a similar way. This vector module is called the **Verma module**. To construct the Verma module, $\mathfrak{so}(2,d)$ is decomposed into the maximal compact subalgebras $\mathfrak{so}(2)\oplus\mathfrak{so}(d)$, spanned by $D$ and $J_{ij}$, respectively, and consider a highest or lowest weight vector $|\Delta,v\rangle$ where $\Delta$ is eigenvalue for $D$ operator and $v$ carries $J_{ij}$ representation index. In this construction, an infinite-dimensional vector module can be construct using the techniques as we explained. To see more details, please see Chap. 8.

### 1.4.7 Casimir Operator

So far, we have discussed about how can we construct the irreducible representations of semisimple groups. For several reasons, it is necessary to distinguish those representations. For instance, quantum fields (or particles) in Minkowski spacetime are irreducible representation of the Poincaré algebra $\mathfrak{iso}(1,d-1)$. How can we distinguish them each other? The answer is simple: mass and spin. Mathematically speaking, the mass and the spin of the fields are given from the Casimir operator of the $\mathfrak{iso}(1,d-1)$, as we will see in Sec. 1.4.9.
The **Casimir operator** (sometimes called **Casimir element** or **Casimir invariant**). can be defined in the **universal enveloping algebra**.

---

**Definition** | **Universal enveloping algebra**

---

A universal enveloping algebra $\mathcal{U}$ for the Lie algebra $\mathfrak{g}$ is written as

$$\mathcal{U}(\mathfrak{g})\equiv\mathcal{T}(\mathfrak{g})/\mathcal{I}\,, \tag{1.4.126}$$

where $\mathcal{T}$ is referred by the **tensor algebra**, given as

$$\mathcal{T}(\mathfrak{g})\equiv\bigoplus_{k=0}^{\infty}\mathfrak{g}^{\otimes k}\,, \tag{1.4.127}$$

and $\mathcal{I}$ is the ideal of $\mathcal{T}(\mathfrak{g})$ such that

$$\mathcal{I}=\{X\otimes Y-Y\otimes X-[X\,,Y]\}\,, \tag{1.4.128}$$

for $X\,,Y\in\mathfrak{g}$.

For example, let us consider $\mathfrak{g}=\mathfrak{su}(2)=\mathrm{Span}\,\{J_z\,,J_+\,,J_-\}$. Since $\mathfrak{g}$ is a vector space, new vectors obtained from the linear combinations of the generators, e.g., $J_z+3J_++2J_-$,

is also an element of $\mathfrak{g}$. On the other hand, multiplied quantities such as $J_z^2$, is not an element of $\mathfrak{g}$ because the only the multiplication between two Lie algebra element is given by the Lie bracket. But the universal enveloping algebra provides "usual" notion of the multiplication as the tensor product, that is, $J_z^2 \in \mathcal{T}(\mathfrak{su}(2))$. In addition, $\mathcal{I}$ implements the Lie bracket of $\mathfrak{g}$ in $\mathcal{T}(\mathfrak{g})$, so two elements of $\mathcal{T}(\mathfrak{g})$ are equivalent up to $\mathcal{I}$.
Now we define the **Casimir operator**.

---

**Definition** | **Casimir operator**

---

A (quadratic) Casimir operator $\mathcal{C}_2 \in \mathcal{U}(\mathfrak{g})$ is defined as

$$\mathcal{C}_2 \equiv \kappa^{ab} T_a T_b \,, \tag{1.4.129}$$

where $T_a$ is a generator of $\mathfrak{g}$ and $\kappa^{ab}$ is the Killing form of $\mathfrak{g}$.

From this definition, one can easily prove that $\mathcal{C}_2$ is commuting with all other generators, namely, $[\mathcal{C}_2 \,, T_a] = 0 \,,\ \forall T_a$. This property is valuable due to the **Schur's lemma**.
Lemma: Any intertwining operator between two *irreducible representation* $\rho_V$ on $V$ and $\rho_W$ on $W$ is either $0$ or invertible.

**Proof** First an intertwining operator $\phi: \ V \to W$ satisfies

$$\phi \cdot \rho_V = \rho_W \cdot \phi \,. \tag{1.4.130}$$

In other word, for $v \in V$ and $X \in \mathfrak{g}$,

$$\phi(\rho_V(X) \cdot v) = \rho_W(X) \cdot \phi(v) \,. \tag{1.4.131}$$

From this definition, $\operatorname{Ker}\phi$ is an invariant subspace of $V$ since $\rho_V(X) \cdot v$ belong to $\operatorname{Ker}\phi$ for $v \in \operatorname{Ker}\phi$. Moreover, $\operatorname{Im}\phi$ is an invariant subspace of $W$. But we assume the irreducibility of $V$ and $W$, so there are only two options: $\operatorname{Ker}\phi = \{0\}$ and $\operatorname{Im}\phi = W$, or $\operatorname{Ker}\phi = V$ and $\operatorname{Im}\phi = \{0\}$ First, $\operatorname{Ker}\phi = \{0\}$ and $\operatorname{Im}\phi = W$. This implies that $\phi$ is an invertible map, so that it is an isomorphism. On the other hand, when $\operatorname{Ker}\phi = V$ and $\operatorname{Im}\phi = \{0\}$, then $\phi$ is zero map. ■
As consequences of Schur's lemma, discussed in Sec.1.3, $\mathcal{C}_2$ is proportional to the identity on an irreducible representation of $\mathfrak{g}$. We will show this statement. For an irreducible represnetastion $\rho$ with $V$, $\rho(\mathcal{C}_2)$ satisfies

$$[\rho(\mathcal{C}_2) \,, \rho(X)] = 0 \,, \forall X \in \mathfrak{g} \,. \tag{1.4.132}$$

Rephrasing the left-handed side as the commutator, we find

$$\rho(\mathcal{C}_2) \cdot \rho(X) = \rho(X) \cdot \rho(\mathcal{C}_2) \,, \tag{1.4.133}$$

and this implies that $\rho(\mathcal{C}_2)$ is an intertwiner between $V$ and $V$. If $\rho(\mathcal{C}_2)$ has an eigenvalue $\lambda$,

$$\rho(\mathcal{C}_2) - \lambda \mathbb{1} \tag{1.4.134}$$

is still an intertwiner. But clearly, it is not invertible, so it is not an intertwiner unless it is a zero map. So, $\rho(\mathcal{C}_2) = \lambda \mathbb{1}$. Since the Casimir operator is proportional to the identity in a certain irreducible representation, irreducible representations can be distinguished from the value of the Casimir operator $\lambda$.
Note that a highest weight representation can be understood from the universal enveloping algebra. Let $v_\lambda$ suppose to be a highest weight vector, that is, it satisfies

$$\rho(E_\alpha)\, v_l = 0\,, \alpha \in \Phi_+\,, \quad \rho(H_\alpha)\, v_\lambda = (\alpha\,, \lambda)\, v_\lambda\,, \tag{1.4.135}$$

then we can find string of the vectors $V_\lambda$, given as

$$V_\lambda = \{\rho^{n_{\alpha_1}}(E_{-\alpha_1})\, \rho^{n_{\alpha_2}}(E_{-\alpha_2}) \,\cdots\, \rho^{n_{\alpha_r}}(E_{-\alpha_r})\, v_\lambda\}\,, \tag{1.4.136}$$

where $n_{\alpha_k} \in \mathbb{N}$ and $\alpha_i \in \Phi_+$. This string of vectors is obviously given as the tensor product of the universal enveloping algebra and the highest weight vector, namely,

$$V_\lambda \simeq \mathcal{U}(n_-) \otimes v_\lambda\,, \tag{1.4.137}$$

where $\mathfrak{n}_-$ is the sum of root spaces with negative roots,

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha\,. \tag{1.4.138}$$

If you want to study more applications of the universal enveloping algebra, please see [7].

### 1.4.8 Young Diagram

### 1.4.9 Representation Theory of Poincaré Group