Lectures on Renormalisation & Critical Phenomena

I. Introduction

Maxwell theory of electrodynamics (1865)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$S = \frac{-1}{4e^2}\int d^4x\,(\vec{E}^2 - \vec{B}^2) \sim -\frac{1}{\alpha}\int d^4x\,(\vec{E}^2 - \vec{B}^2)$$

α: fine-structure const. → Coupling constant

↓ Quantization

$$A_\mu = (\phi, \vec{A}), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

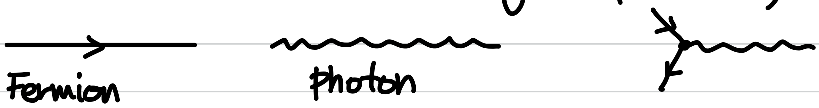
$$S = -\frac{1}{4e^2}\int d^4x\,(F_{\mu\nu}F^{\mu\nu} - \underbrace{A_\mu J^\mu}_{\text{Current}}) + (\text{matter term})$$

(matter term) → Schrödinger eqn. / Dirac eqn.

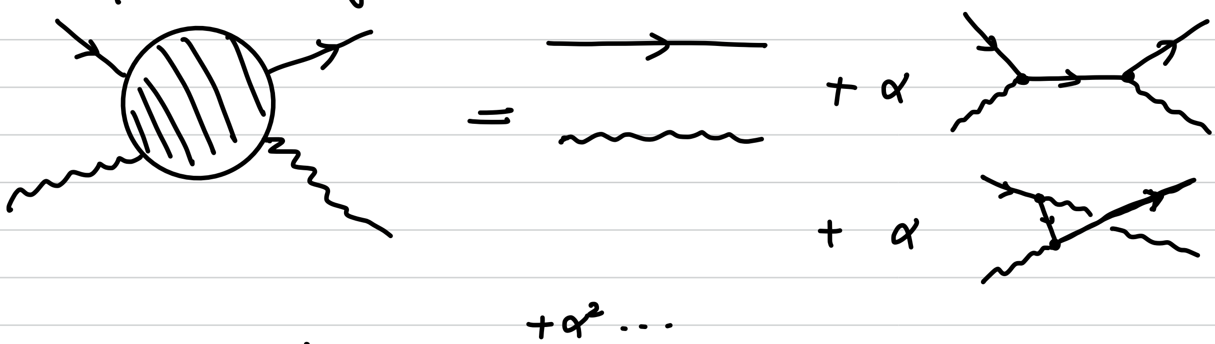
If we consider Fermion case,

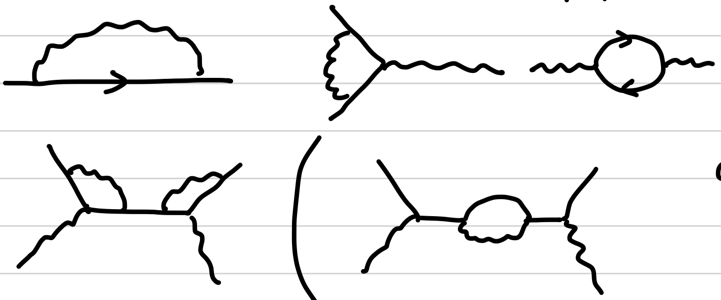
$$S = \int d^4x\left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu + iA_\mu - m)\psi\right]$$

⇒ We can calculate scattering amplitude by this Lagrangian.



i.e. Compton scattering





Loop diagrams will contribute in the higher order terms, and these diagrams are diverged.

$$\sim \int dq^4\,\frac{1}{q^4} \sim [\log q]_0^\infty \Rightarrow \text{Divergent integral}$$

Kramers, Bethe, Schwinger, Feynman, Tomonaga, Dyson 1947-49

Renormalisation

<u>idea</u> Extract infinites from the Lagrangian by renormalise all coupling constants and fields.

$$\psi_B = Z_2^{1/2}\psi_R \qquad A_\mu^B = Z_3^{1/2} A_\mu^R \qquad m_B = Z_m m_R \qquad e_B = Z_e e_R$$

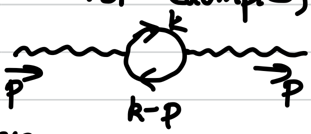
$$Z_i = 1 + \delta_i$$

$\delta_i \to$ Counterterm
kill diverging scattering amplitude

$\Rightarrow$ In this context, physical quantities are just substraction between two infinities.

$\Rightarrow$ Also, it is natural to introduce two new energy scale, called cut-off Λ and renormalisation scale μ. As the consequence of renormalisation, all these parameters become functions of μ. For example,

$$e_{eff}^2(-p^2) = e_{eff}(\mu)^2 \left[1 + \frac{e_{eff}(\mu)^2}{12\pi^2} \log \frac{-p^2}{\mu^2}\right]$$



As we shown above, coupling "constants" are no longer constant after renormalisation. They depend on the energy scale.

(Kenneth Wilson, 1971) Renormalisation can be interpreted as the coarse-graining in the lattice system!

$\Rightarrow$ Renormalisation group (RG)

I. Ising Model in $d=1$

Description of (ferro-) magnetism by spin lattice system.

$$H(s_i;\{K\}) = -\frac{1}{\beta}\sum_{\{n\}} K^{(n)} \sigma^{(n)}_{i(n)} \qquad \begin{array}{l}\sigma : \text{Local operator} \\ K : \text{coupling constant}\end{array}$$

i.e. $n=1 \quad \sigma^{(1)}_i = s_i \qquad n=2 \quad \sigma^{(2)}_{ij} = s_i s_j \qquad n=3 \quad \sigma^{(3)}_{ijk} = s_i s_j s_k \quad \cdots$

Here, we only consider $n=1$ and $n=2$ for simplicity.

Note Although we only consider a part of the whole theory, the remaining part will be restored from renormalisation, that is, we can find $n>2$ terms after renormalisation of $n=1$ and $n=2$ theory.

$$H(s;J,h) = -h/_{\beta}\sum_{i=1}^{N} s_i - J/_{\beta}\sum_{i=1}^{N-1} s_i s_{i+1}$$

J : exchange interaction $\qquad J>0$: Ferromagnetic Ising

h : Zeeman $\qquad J<0$: Anti- 〃

$$Z(J,h,T) = \underbrace{\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}} e^{-\beta H(s;J,h)} = e^{-\beta F(J,h,T)}$$

Ising model has time-reversal symmetry

i) H has time-reversal symmetry when $h=0$

$$T\vec{s}\,T^{-1} = -\vec{s} \qquad T : \text{time-reversal}$$

$$T H(s;J,h) T^{-1} = H(-s;J,h) = H(s;J,-h)$$

ii) For $h \neq 0$

$$Z(J,h,T) = \sum_{\{s_i\}} e^{-\beta H(s;J,h)}$$

$$= \sum_{\{s_i\}} e^{-\beta H(-s;J,h)} = \sum_{\{s_i\}} e^{-\beta H(s;J,-h)} = Z(-h,J,T)$$

$\Rightarrow$ "Effectively" it is invariant under $h \to -h$.

- 1D Ising model

One of the useful description for magnetic system : magnetisation

$$m = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle$$

For $h \approx 0$, the ground state is given

$\cdots \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \cdots \qquad E = -NJ \quad S = k_B \log\Omega = 0$

$F_0 = E - TS = -NJ, \quad m=1$

First excited state : $\qquad$ Single domain wall

$\cdots \uparrow\uparrow\uparrow\uparrow \,\vdots\, \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \cdots \qquad E = (N-1)(-J) + J = -J(N-2)$

$S = k_B \log N, \quad m = 0$

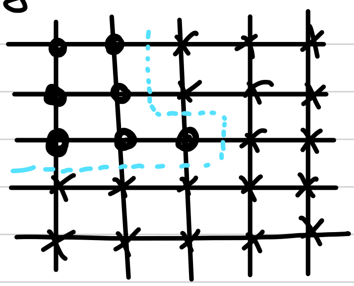
$F_1 = -J(N-2) - k_B T \log N$

$$\Delta F = F_1 - F_0 = [-J(N-2) - k_B T\log N] - [-NJ]$$

$$= 2J - k_B T \log N$$

In thermodynamic limit $N \to \infty \Rightarrow \Delta F < 0$
$\Rightarrow$ No phase transition at finite temperature.

- 2d Ising model



n : misaligned bond

$\Delta E = 2J \times n$

$\Delta S \simeq k_B \log 3^n$ (in general $k_B \log (z-1)^n$) coord. #

$$\Delta F = \Delta E - T\Delta S$$
$$= 2Jn - nk_B T \log 3$$
$$= n[2J - k_B T \log 3]$$

Roughly: $N \to \infty \Leftrightarrow n \to \infty \Rightarrow \Delta F > 0$ if $2J > k_B T \log 3$

$$\Rightarrow T < \frac{2J}{k_B \log 3} \equiv T_c$$

$$0 \;\big|\!\xrightarrow{\;m\neq 0\;}\!\big|_{T_c}\!\xrightarrow{\;m=0\;} T$$

In short, there exists the long range order for $d > 1$ and we call $d_c = 1$ as the lower critical point.

I.2. Details of d=1 Ising model

$h = 0$

$$H = -J \sum_{i=1}^{N-1} s_i s_{i+1}$$

$s_1 \; s_2 \; s_3 \; \cdots \; s_N$

$$Z = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_N} e^{J \sum^{N-1} \eta_i} \qquad \eta_i = s_i s_{i+1}, \quad i = 1, \cdots, N-1$$

$$= \sum_{\eta_1} \cdots \sum_{\eta_{N-1}} \sum_{s_N} e^{J \sum^{N-1} \eta_i}$$

$$= \sum_{s_N} \Big[\sum_{\eta = \pm 1} e^{J\eta} \Big]^{N-1} = \sum_{s_N} (e^J + e^{-J})^{N-1}$$

$$= 2^N \cosh^{N-1} J(T) = 2^N \cosh^{N-1} \tilde{J}\beta$$

In thermodynamic limit : $N \to \infty$

$$Z[J] = 2^N \cosh^N J = (e^J + e^{-J})^N$$

$$F = -k_B T \log Z = -N k_B T \log (e^J + e^{-J})$$

$$= -N k_B T \left(J + \log (1 + e^{-2J}) \right)$$

More convenient quantity : free energy density

$$f = F/N = \begin{cases} -J/\beta & T \to 0 \\ -k_B T \log 2 & T \to \infty \end{cases}$$

- Internal energy $U = \langle E \rangle = -\frac{\partial}{\partial \beta} \log Z$

$$= -N \frac{\partial}{\partial \beta} \log[2\cosh \tilde{J}\beta]$$

$$= -N k_B T J \tanh(J)$$

- Specific heat $C = \frac{dU}{dT} = -\frac{1}{k_B T^2}\frac{\partial U}{\partial \beta}$

$$= \frac{NJ}{k_B} \operatorname{sech}^2 J$$

- Correlation function

$$G(i,j) = \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle$$

$$= \langle s_i s_j \rangle \qquad \text{since } \langle s_i \rangle = 0$$

$$G(i,i+j) = \frac{1}{Z} \sum_{\{s_i\}} s_i s_{i+j} e^{J_1 s_1 s_2 + \cdots + J_{N-1} s_{N-1} s_N} \Big|_{J_1 = \cdots = J_{N-1} = J}$$

$$G(i,i+1) = \frac{1}{Z} \frac{\partial}{\partial J_i} \Big[\sum_{\{s_i\}} e^{J_1 s_1 s_2 + \cdots + J_{N-1} s_{N-1} s_N} \Big] \Big|_{J_1 = \cdots = J_{N-1} = J}$$

$$= \frac{\partial}{\partial J_i} \log Z[\{J_i\}] \Big|$$

$$= \frac{\partial}{\partial J_i} \log \Big[2^N \prod_{i=1}^{N-1} \cosh J_i \Big] \Big| = \tanh J$$

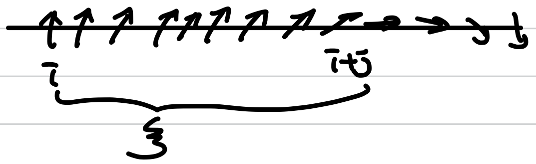
$$G(i,i+2) = \frac{1}{Z} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_{i+1}} Z \Big| = (\tanh J_i)(\tanh J_{i+1}) \Big| = \tanh^2 J$$

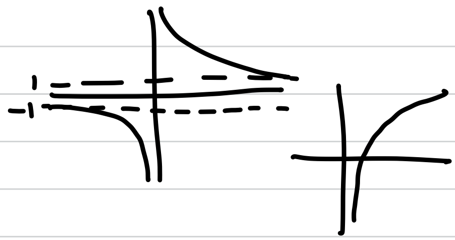
$\hookrightarrow$ $G(i,i+j) = (\tanh J)^j$

If there exists the long range ordering, $G(i,i+j) = 1$ for any j.

Let $G(i,i+j) = e^{-j/\xi}$ $\quad \xi = \frac{1}{\log \coth J}$: correlation length

ξ : approximate length for sustaining i^{th} spin information.





For long range order : $G \to 1$, $\xi \to \infty$

$T \to 0 \quad \xi \to \infty$

For finite T : No divergence on ξ

No long range ordering.

III. Curie-Weiss MFT and Critical Phenomena

III.1. Weiss model

$$H = -J\sum s_i s_j - h\sum s_i$$

Suppose, $J=0$ (paramagnetism)

$$\mathcal{Z}[0,h,T] = \prod_{i=1}^{N}\left(e^{h} + e^{-h}\right) = (2\cosh h)^N$$

$$m = -\frac{1}{N}\frac{\partial F}{\partial h} = \tanh h = \tanh\frac{h}{k_BT}$$

When $J\neq 0$, each spin feels not only the external field h, but also an effective field h_{eff} due to the all other spins.

The main problem is that h_{eff} is unknown

$$h_i^{eff} = h + \sum_j J_{ij} s_j = h + \sum_j J_{ij}\underbrace{\langle s_j\rangle}_{\text{mean field}} + \cancel{\sum_j J_{ij}\underbrace{(s_j - \langle s_j\rangle)}_{\text{fluctuation}}}$$

ignored in MFT

$$= h + 2dJm$$

d-dim hypercube

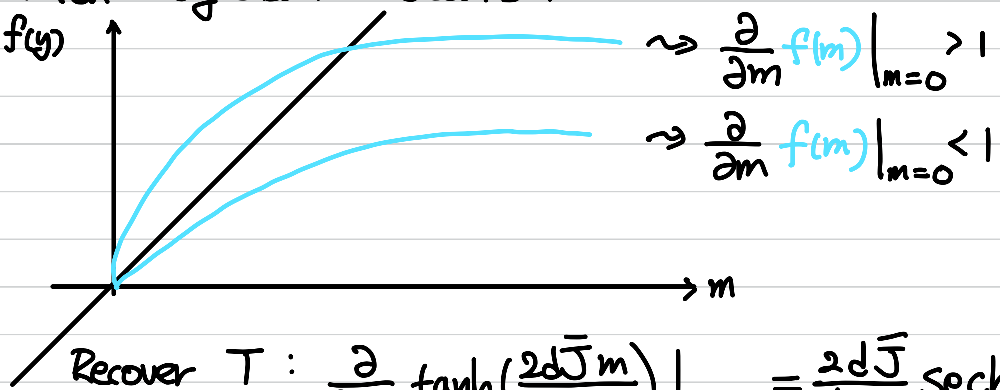
$$\therefore\ m = \tanh h_{eff} = \tanh(h + 2dJm)$$

$\Rightarrow$ We can determine m and consequently h_{eff} in self-consistent way.

Guess m $\Rightarrow$ evaluate h_{eff} $\Rightarrow$ find m $\Rightarrow$ $\cdots$

Setting $h=0$ $\qquad m = \tanh(2dJm)$

When magnetisation occurs?



Recover T: $\displaystyle \frac{\partial}{\partial m}\tanh\left(\frac{2dJm}{k_BT}\right)\Big|_{m=0} = \frac{2dJ}{k_BT}\operatorname{sech}^2\left(\frac{2dJm}{k_BT}\right)\Big|_{m=0}$

$$= \frac{2dJ}{k_BT} \equiv \frac{T_c}{T} \equiv \tau$$

If $T < \frac{2dJ}{k_B}$, two curves are intersecting both $m=0$ and $m>0$

$\Leftrightarrow \tau > 1$

If $T > \frac{2dJ}{k_B}$, two curves are only intersecting at $m=0$.

$\Leftrightarrow \tau < 1$

We say that $T_c = \frac{2dJ}{k_B}$ is critical temperature where phase transition occurs.

III.2. First Look at the Critical Exponents.

If we turn on the Zeeman term,

$$m = \tanh(h + m\tau) = \frac{\tanh h + \tanh m\tau}{1 + \tanh h \tanh m\tau}$$

$$\Rightarrow \tanh h = \frac{m - \tanh m\tau}{1 - m \tanh m\tau}$$

For small h and m

$$\left(h - \frac{h^3}{3} + \cdots\right) = \frac{m - m\tau + \frac{(m\tau)^3}{3} - \cdots}{1 - m\left(m\tau - \frac{(m\tau)^3}{3} + \cdots\right)}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$$

$$\frac{1}{1-x} \simeq 1 + x + x^2 + \cdots$$

$$= \left(m - m\tau + \frac{(m\tau)^3}{3} - \cdots\right)\left(1 + m\left(m\tau - \frac{(m\tau)^3}{3} + \cdots\right)\right)$$

$$= \left(m - m\tau + \frac{m^3}{3}\tau^3 - \cdots\right) + m^3\tau - m^3\tau^2 + \cdots$$

$$h \simeq m - m\tau + m^3\left(\tau - \tau^2 + \frac{\tau^3}{3} + \cdots\right)$$

For $h = 0$ and $\tau \Rightarrow 1 + \epsilon \Leftrightarrow T \to T_c - \epsilon$

$$0 \simeq m - m(1+\epsilon) + m^3\left((1+\epsilon) - (1+2\epsilon) + \frac{1}{3}(1+3\epsilon)\right)$$

$$= -m\epsilon + m^3\left(\frac{1}{3}\right)$$

$$m^2 \simeq 3\epsilon \simeq 3(\tau - 1) = 3\frac{T_c - T}{T} \simeq 3\frac{T_c - T}{T_c}$$

$$\Rightarrow m^2 \simeq 3\frac{T_c - T}{T_c} + \cdots \quad \text{or} \quad m \simeq \left(3\frac{T_c - T}{T_c}\right)^{1/2}$$

$$\simeq [3(\tau - 1)]^{1/2}$$

- We call the exponent $\frac{1}{2}$ as the critical exponent.

Critical exponents tell us critical behaviour of the system near the critical point.

III.3. Critical Phenomena of the Curie-Weiss

Phase : A region where free energy density f is analytic.

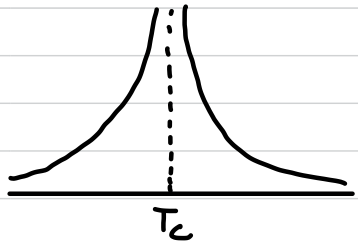
Critical point : The point in the phase diagram where a continuous phase transition occurs

Critical exponent : Exponent of reduced temperature $1 - \tau \equiv t$ near the critical point.

Why critical exponents are important?

- They indicates universality class which gives a hint for underlying UV theory.
- They are strongly related with the scaling law

① Specific heat



$$C_V = -T \frac{\partial^2 f}{\partial T^2}\Big|_{h=0} = C_\pm |t|^{-\alpha}$$

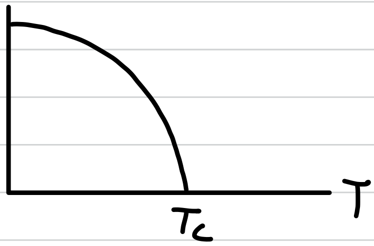
$$= \frac{(2d\bar{J}m)^2}{k_B T^2} \operatorname{sech}^2\left(\frac{2d\bar{J}m}{k_B T}\right)$$

$$= m^2 k_B \frac{T_c^2}{T^2} \operatorname{sech}^2\left(\frac{T_c}{T} m\right) = \left(\frac{1}{1+t}\right)^2 m^2 k_B \operatorname{sech}^2\left(\frac{m}{1+t}\right)$$

$\Rightarrow$ It does not diverge when $t > 0$. This fact implies $\alpha = 0$.

$\Rightarrow \boxed{\alpha = 0}$

② Magnetisation at $h=0$



$$m = -\left(\frac{\partial f}{\partial \bar{h}}\right)_{h=0} = m_-(-t)^\beta \qquad \beta > 0$$

$$\boxed{\beta = \frac{1}{2}}$$

③ Isothermal magnetic susceptibility

$$\chi = -\frac{\partial^2 f}{\partial \bar{h}^2}\Big|_{h=0} = \chi_\pm |t|^{-\gamma} \quad \text{“} = \frac{\partial m}{\partial h}\text{”}$$

$$h \simeq m - m\tau + m^3\left(\cancel{\tau - \tau^2} + \frac{\tau^3}{3} + \cdots\right)$$

$$m = m\tau + \bar{h}\beta - \frac{m^3}{3}\tau^3 \Rightarrow \chi = \chi\tau + \beta - \chi m^2 \tau^3$$

$$\chi = \frac{\beta}{1 - \tau + m^2\tau^3} \qquad \tau = 1 - t, \quad m^2 \simeq -3t \text{ near } t \simeq 0^-$$

$$\simeq \frac{\beta}{t - 3t(1-t)^3} \simeq -\frac{\beta}{2}|t|^{-1} \Rightarrow \boxed{\gamma = 1}$$

④ Magnetisation at $T = T_c$ (or $t = 0$)

$$m(h) \propto |h|^{1/\delta}$$

$$\operatorname{arctanh} m = \beta(2d\bar{J}m + \bar{h})$$

$$(2d\bar{J}m + \bar{h}) \simeq k_B T\left(m + \frac{m^3}{3} + \frac{m^5}{5} + \cdots\right)$$

$$\Rightarrow \bar{h} \simeq m k_B (T - T_c) + \frac{m^3}{3} k_B T + \cdots$$

$$T = T_c \quad \simeq \frac{m^3}{3} k_B T_c \qquad \Rightarrow \boxed{\delta = 3}$$

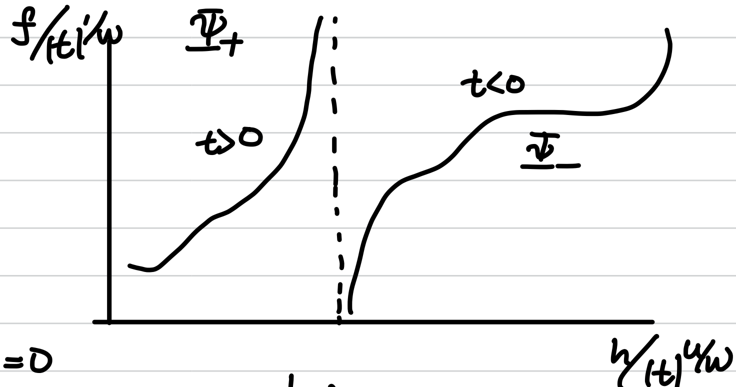
II.4. The Scaling Law

Where these critical exponents are coming from?

Hypothesis: Critical exponents are a consequence of the scaling free energy.

<u>Ansatz</u> $\quad f(T, h) = |t|^{1/w} \Psi_\pm\left(\frac{h}{|t|^{y/w}}\right)$

where Ψ is discontinuous near $t=0$.



① $C_v = -T \frac{\partial^2 f}{\partial T^2}\Big|_{h=0} = -\frac{T}{T_c^2}\frac{\partial^2 f}{\partial t^2}$

$-T_c C_v = \frac{\partial^2 f}{\partial t^2}$

$= \frac{\partial}{\partial t}\left[\frac{1}{w}|t|^{\frac{1}{w}-1}\Psi_\pm + |t|^{\frac{1}{w}}\left(-\frac{u}{w}\right)\frac{h}{|t|^{u/w+1}}\Psi_\pm\right]_{h=0}$

$= \frac{1}{w}\left(\frac{1}{w}-1\right)|t|^{\frac{1}{w}-2}\Psi_\pm(0) + \cancel{h(\cdots)} \to 0 \quad = \frac{1}{w}\left(\frac{1}{w}-1\right)|t|^{\frac{1}{w}-2}\Psi_\pm(0)$

$= -T_c C_\pm |t|^{-\alpha}$

$\Rightarrow T_c C_\pm = -\frac{1}{w}\left(\frac{1}{w}-1\right)\Psi_\pm(0), \quad \alpha = 2-\frac{1}{w}$

② $m = -\frac{\partial f}{\partial h}\Big|_{h=0}$

$= -|t|^{\frac{1}{w}-\frac{u}{w}}\Psi'_\pm(0) = m_-|t|^\beta$

$\Rightarrow m_- = -\Psi'_\pm(0), \quad \beta = \frac{1}{w}-\frac{u}{w}$

③ $\chi = -\frac{\partial^2 f}{\partial h^2}\Big|_{h=0} = -|t|^{\frac{1}{w}-\frac{2u}{w}}\Psi''_\pm(0) = \chi_\pm(0)|t|^{-\gamma}$

④ Let $x \equiv h/|t|^{u/w}$

$f(t,h) = \left(\frac{h}{x}\right)^{\frac{1}{u}}\Psi_\pm(x)$

$= h^{\frac{1}{u}}\frac{1}{x^{\frac{1}{u}}}\Psi_\pm(x) \equiv h^{\frac{1}{u}}\Phi(x)$

$m = -\frac{\partial f}{\partial h} = -\frac{1}{u}h^{\frac{1}{u}-1}\Phi_\pm(x) + h^{\frac{1}{u}}\frac{1}{|t|^{u/w}}\Phi'_\pm(x)$

$h\to 0 \quad \simeq -\frac{1}{u}h^{\frac{1}{u}-1}\Phi_\pm(\infty) \propto h^{1/\delta}$

$\Rightarrow \delta = \frac{u}{1-u}$

$\alpha = 2-\frac{1}{w} \quad \beta = \frac{1-u}{w} \quad \gamma = \frac{2u-1}{w} \quad \delta = \frac{u}{1-u}$

$\Rightarrow$ Since only two exponents are independent, we can derive the scaling law.

III. 5. Scaling Law

i) $\alpha + 2\beta + \gamma = 2$: Rushbrooke's law

ii) $\alpha + \beta(\delta+1) = 2$: Griffith's scaling law

- Correlation function

<u>Recall</u>

$$G(r_i - r_j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

$$\begin{aligned}
\chi = \frac{\partial m}{\partial h}\Big|_{h\to 0} &= \frac{\partial}{\partial h}\left[\frac{1}{N\beta}\frac{1}{Z}\frac{\partial Z}{\partial h}\right] \\
&= \frac{1}{N\beta}\left\{\frac{1}{Z}\frac{\partial^2 Z}{\partial h^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial h}\right)^2\right\} \\
&= \frac{\beta}{N}\left[\sum_{ij}\langle s_i s_j\rangle - \left(\sum_i \langle s_i\rangle\right)^2\right] \\
&= \frac{\beta}{N}\sum_{ij} G(r_i - r_j) = \frac{\beta}{N}\sum_j \sum_{x_i} G(x_i) \\
&= \beta \sum_{x_i} G(x_i)
\end{aligned}$$

$$\chi(T) = \beta \sum_{x_i} G(x_i) \Rightarrow \chi(T) = \beta \int \frac{d^d r}{a^d} G(\vec{r}) \quad , \quad a: \text{lattice constant}$$

<u>Ansatz</u> $G(\vec{r}, t) = \frac{\Phi(r/\xi(t))}{r^{d-2+\eta}}$, $\xi(t) \propto |t|^{-\nu}$ ν: critical exponent for correlation length

η: anomalous dimension

$$\chi(T) \propto \int d^d r\, G(\vec{r}, t) \overset{x = r/\xi}{=} \xi^d \int d^d x \frac{\Phi(x)}{\xi^{d-2+\eta} x^{d-2+\eta}}$$

$$= \xi^{2-\eta} \int d^d x \frac{\Phi(x)}{x^{d-2+\eta}}$$

$$\Rightarrow \chi(T) \propto \xi^{2-\eta} \propto |t|^{-\nu(2-\eta)} \propto |t|^{-\gamma}$$

$$\Rightarrow \gamma = \nu(2-\eta) \quad : \text{Fischer's scaling law}$$

Let us supposed to be that ξ is the only relevant length scale near $t \sim 0$.

$$\Rightarrow f \propto \xi^{-d} \sim |t|^{\nu d}$$

$$C_v \propto \frac{\partial^2 f}{\partial t^2} \propto |t|^{\nu d - 2} \propto |t|^{-\alpha} \Rightarrow \boxed{\alpha = 2 - \nu d}$$ Josephson's scaling law

Hyperscaling

: Critical exponent is depending on the dimensionality of the system.

* Hyperscaling can be violated when the mean field theory is valid due to the dangerously irrelavant variables.

We call the dimension when the mean field theory is valid as the upper critical dimension d_u.

But how we trust / ensure that these scaling assumptions are correct?

$\Rightarrow$ Renormalisation!

IV. Landau Theory for Phase Transition

IV.1. Assumptions

Let f_L is the free energy functional with respect to the coupling constants $\{k_i\}$ (i.e. h, J in Ising) and order parameter η (i.e. m in Ising).

Then f_L should be satisfying:

(i) f_L obeys the symmetry of the system.

(ii) Near T_c, f_L is an analytic function for both $\{k_i\}$ and η.

$$f_L = \sum_{n=0}^{\infty} a_n([K])\eta^n$$

<u>Notes</u>
- Free energy functional is an effective Hamiltonian of the system. It is a consequence of the renormalisation of the "original" theory (or UV theory)
- The highest order must be even and its coefficient must be positive.
- Validity of the Landau theory is depending on the dimensionality of the theory.
- Later, we will handle the order parameter η as a function of the space (or Euclidean spacetime) $\eta(\vec{r})$.
- Here, we will deal with the (reduced) ~~temperature~~ t as a kind of the coupling constant, that is, $a_n = a_n(t, [K])$.
- If the system is not a uniform one, f_L can contain the derivative terms (i.e. $\partial_x \eta$).

IV.2. Landau Theory of the Ising Model.

For simplicity we set that $h=0$ case.

Then, this system has the time-reversal symmetry $s \to -s$, so f_L is symmetric under $\eta \to -\eta$, i.e. $f_L(-\eta) = f_L(\eta)$

$$\Rightarrow f_L = a_0 + a_2\eta^2 + a_4\eta^4 + \mathcal{O}(\eta^6)$$

- We set $a_0 = 0$ since it is nothing but the energy shift.
- $a_4 > 0$ as we discussed before.

$$\Rightarrow f_L = a\eta^2 + b\eta^4$$

Condition for minimum:

$$\frac{\partial f_L}{\partial \eta} = 0 = 2a\eta_* + 4b\eta_*^3$$

$$\Rightarrow \eta_* = 0 \quad \text{or} \quad \eta_* = \sqrt{-\frac{a(t)}{2b}}$$

If $a(t) \sim a_0 + a_1 t + \mathcal{O}(t^2)$

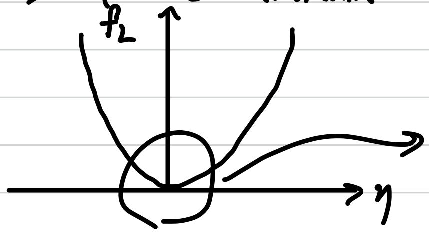
$$\Rightarrow \eta(t=0) = 0 \Rightarrow a_0 = 0$$

$$\Rightarrow f_L = at\eta^2 + bt^4, \quad a, b > 0$$

When the external field is applied, $f_L = -h\eta + at\eta^2 + b\eta^4$

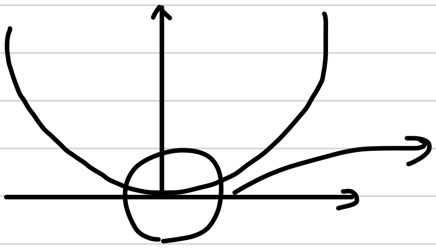
$h=0$

① $t>0 \Rightarrow$ Global minimum occures at $\eta_*=0$



$f_L \sim \eta^2$

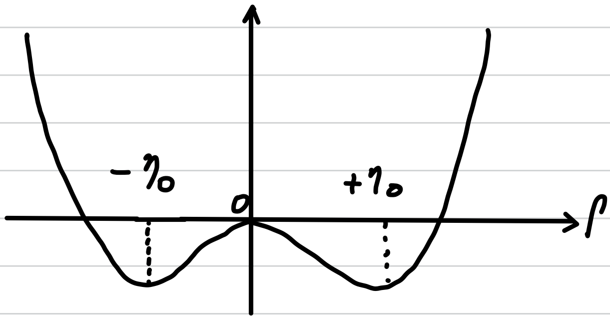
② $t=0$



$f_L \sim \eta^4$

③ $t<0 \Rightarrow \eta_* = 0$ or $\pm\sqrt{-\frac{at}{2b}}$

$\eta_0 = \sqrt{-\frac{at}{2b}}$



- $\eta=0$ is not only no longer the global minimum, but also the local maximum.
- There exist double minimum related to the symmetry $\eta \to -\eta$.

$\Rightarrow$ Spontaneous symmetry breaking

IV.3. Critical Exponents for Landau Theory

$$f_L = -h\eta + at\eta^2 + b\eta^4$$

② $\left.\frac{\partial f_L}{\partial \eta}\right|_{h=0} = 0 \Rightarrow \eta^2 = -\frac{a}{2b}t \Rightarrow \beta = \frac{1}{2}$

① Specific heat

$$f_L = at\left(-\frac{a}{2b}t\right)^2 + b\left(-\frac{a}{2b}t\right)^4 = -\frac{a^2}{4b}t^2 \quad (t<0)$$

$$C_V = -\frac{T}{T_c^2}\frac{\partial^2 f_L}{\partial t^2} = -\frac{T}{T_c^2}\left(-\frac{a^2}{2b}\right) \quad t<0$$

$$\Rightarrow C_V = \begin{cases} 0 & t>0 \\ \frac{a^2}{2bT_c} & t<0 \end{cases} \Rightarrow \alpha = 0 \text{ (discontinuous)}$$

④ $\frac{\partial f_L}{\partial \phi} = 0 \Rightarrow h = 2at\eta + 4b\eta^3 \quad (*)$

at $t=0 \Rightarrow h \sim \eta^3 \Rightarrow \delta = 3$

③ Differentiating (*) both sides by h

$$1 = 2at\frac{\partial \eta}{\partial h} + 12b\frac{\partial \eta}{\partial h}\eta^2$$

$$\chi = \frac{\partial \eta}{\partial h} = \frac{1}{2at+12b\eta^2} \qquad \eta^2 = \begin{cases} 0 & t>0 \\ -\frac{at}{2b} & t<0 \end{cases}$$

$$= \begin{cases} \frac{1}{2at} & t>0 \\ \frac{1}{-4at} = \frac{1}{4a|t|} & t<0 \end{cases} \quad \propto |t|^{-1} \Rightarrow \gamma = 1$$

IV.3. First Order Transition

More general free energy form

$$f_L = at\eta^2 + c\eta^3 + b\eta^4$$

$$\frac{\partial f_L}{\partial \eta} = 0 = 2at\eta + 3c\eta^2 + 4b\eta^3 \qquad 4\times(4b)\times(2a) =$$

$$= \eta\,(4b\eta^2 + 3c\eta + 2at) = 0$$

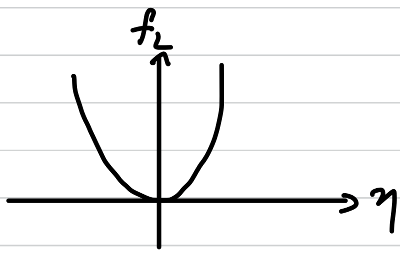
$$\Rightarrow \eta = 0 \quad \text{or} \quad \eta = \frac{-3c \pm\sqrt{9c^2-32abt}}{8b} = -\frac{3c}{8b} \pm \sqrt{\left(\frac{3c}{8b}\right)^2 - \frac{at}{2b}}$$

To be $\eta_* \neq 0 \Rightarrow \left(\frac{3c}{8b}\right)^2 \geq \left(\frac{at}{2b}\right) \Rightarrow t_* \leq \frac{9c^2}{4ab}$

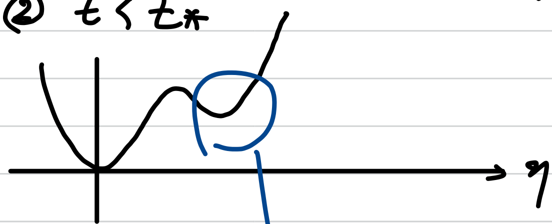
Since $t_* > 0$, a kind of the phase transition can occur over the critical temperature $t=0$.

$$12b\eta^2 + 6c\eta + 2at = 0$$

① $t > t_*$

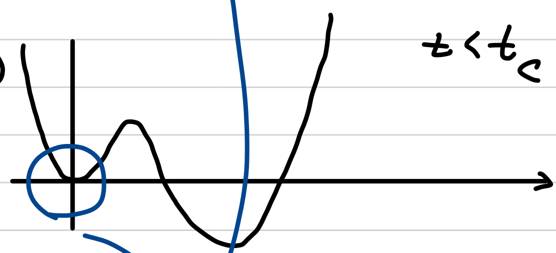


② $t < t_*$

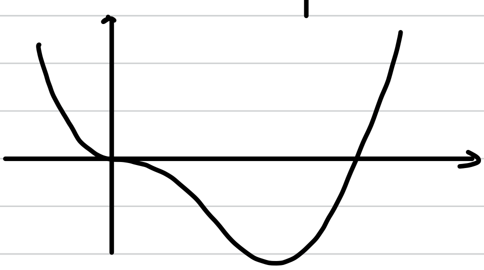


③ $t = t_c$

④ $t < t_c$



⑤ $t = 0$



meta-stable states

$$f_L = at\eta^2 + c\eta^3 + b\eta^4$$

At $t=t_c$, $\eta = \eta_c$, then $f(\eta_c)$ and $\partial f(\eta_c) = 0$

$$\Rightarrow \begin{cases} \eta_c^2[at_c + c\eta_c + b\eta_c^2] = 0 \\ \eta_c[2at_c + 3c\eta_c + 4b\eta_c^2] = 0 \end{cases}$$

$$\Rightarrow c\eta_c + 2b\eta_c^2 = 0 \qquad \therefore \eta_c = -\frac{c}{2b}$$

$$at_c - \frac{c^2}{2b} + b\frac{c^2}{4b^2} = 0 \Rightarrow t_c = \frac{c^2}{4ab}$$

i) When the system is being cooled, $\eta = 0$ is meta-stable and it is called supercooled state.
ii) When the system is being heated, $\eta \neq 0$ is meta-stable and it is called superheated state.

IV.4. Landau - Ginzburg Criterion

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad s_i = \langle s_i \rangle + s_i - \langle s_i \rangle$$

$$= \sum_{\langle ij \rangle} J_{ij} \left(\langle s_i \rangle + s_i - \langle s_i \rangle\right)\left(\langle s_j \rangle + s_j - \langle s_j \rangle\right)$$

$$= \sum_{\langle ij \rangle} J_{ij} \left(\underbrace{\langle s_i \rangle\langle s_j \rangle}_{\text{Mean field}} + \underbrace{s_i s_j - \langle s_i \rangle\langle s_j \rangle}_{\text{Error of MFT}}\right)$$

$$\hookrightarrow E_{ij} \equiv \frac{|\langle s_i \rangle\langle s_j \rangle - \langle s_i s_j \rangle|}{\langle s_i \rangle\langle s_j \rangle} \ll 1 \Rightarrow \text{MFT is reasonable.}$$

fractional error

if not $E_{ij} > 1 \Rightarrow$ We can predict larger system from the smaller one.

$\Downarrow$ in continuous limit

$$E_R = \frac{|G(R)|}{\eta^2} \ll 1 \qquad R \sim |r_i - r_j|$$

$$= \frac{|\int_V d^d r\, G(r)|}{\int_V d^d r\, \eta^2(r)} \qquad \text{Landau - Ginzburg Criterion}$$

For Ising, $f = at\eta^2 + \frac{1}{2}b\eta^4$

$$\frac{\partial f}{\partial \eta} = 0 = 2\eta(at + b\eta^2) \Rightarrow \eta^2 = -\frac{a}{b}t = \frac{a}{b}|t| \quad \text{: ordered state}$$

i) $\int_V d^d r\, \eta^2(r) = \frac{a}{b}|t|\xi^d \propto \frac{a}{b}|t|^{1-\frac{1}{2}d}$ $\qquad \xi \propto |t|^{-1/2}$

$V \to \xi^d$

ii) $\int_V d^d r\, G(r) \propto \chi \propto \frac{1}{4a|t|}$

$$\Rightarrow E_{LG} \propto \frac{1}{|t|^{2-\frac{1}{2}d}} \ll 1$$
$$= \frac{|t_{LG}|^{2-\frac{1}{2}d}}{|t|^{2-\frac{1}{2}d}} \quad \Bigg] \Rightarrow |t|^{\frac{4-d}{2}} \gg |t_{LG}|^{\frac{4-d}{2}}$$

$d>4$: Landau theory is valid $\Rightarrow$ MFT is valid

$d<4$: Landau theory is not valid.

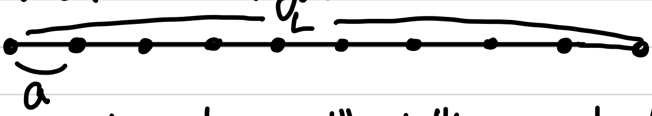
$\underbrace{d=4}$: Marginal. $\Rightarrow$ Logarithmic correction $\chi \sim \frac{1}{t} |\log t|^{1/3}$

↳ upper critical dimension

V. Functional Integral (≈ Path Integral)

V.1. Why functional integral?

Ising



Spin system with lattice constant a

↳ fundamental length scale $\Rightarrow a$

$\Downarrow$

energy scale $\Rightarrow \frac{1}{a} \equiv \Lambda$

Remark $c = \hbar = 1$

$$E = \frac{hc}{\lambda} = \frac{1}{k}$$

We are interested in the universality of the system.
Universality is originated from the UV theory, that is, $a \to 0$ limit.

$$Z = \sum_{\{s_i\}} e^{-\beta H} = \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} e^{-\beta H[s]}$$

all possible "spin" configuration

continuous s

$$\Rightarrow \int ds_0 \int ds_{0+a} \cdots \int ds_{0+Na}\, e^{-\beta H[s]}$$

field
wavefunction

$a \to 0$, fix L

$$\Rightarrow \lim_{N\to\infty} \left[\prod_{n=0}^{N} \left(\int ds_{n \cdot \frac{L}{N}} \right) \right] e^{-\beta H} \;\text{"}\equiv\text{"} \int \mathcal{D}s \; e^{-\beta H}$$

functional integral.

$\Downarrow$ consider all "spin" configurations in the system.

Note The functional integral has ∞ value.
But, all physical observables are given from the ensemble average so this fact is not important. (or just $N \equiv Z^{-1}$)
In the final step, H is actually Landau free energy.

V.2. Formal Derivation of the Path Integral

propagator ⟵ transfer matrix

Recall transfer matrix for Ising model

$$H = -h\sum_i s_i - J\sum_i s_i s_{i+1} \quad \text{with PBC}$$

$$= -h\sum_i \frac{1}{2}(s_i + s_{i+1}) - J\sum_i s_i s_{i+1} \equiv \sum_i \overbrace{\left[-\frac{h}{2}(s_i + s_{i+1}) - J s_i s_{i+1} \right]}^{\mathcal{H}_i}$$

$$Z = \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} e^{-\beta \sum_i \mathcal{H}_i} = \underbrace{\left(\sum_{s_1 = \pm 1} e^{-\beta \mathcal{H}_1} \right)}_{T_{12}} \cdots \underbrace{\left(\sum_{s_N = \pm 1} e^{-\beta \mathcal{H}_N} \right)}_{T_{N1}}$$

T_{ij} : (i,j) entry of the T matrix

↳ T_{ij} represents the "bond" between ith and jth site.

$$= (T_{12}) \cdots (T_{N1}) = \mathrm{tr}\,[T^N]$$

Consider $H = \frac{p^2}{2m} + V(x)$

break time interval N times $\Delta t = \frac{t_f - t_i}{N}$

$$U(t_f ; t_i) = \langle x_f | e^{-i(t_f - t_i)H} | x_i \rangle$$

$$= \int dx_N \cdots \int dx_1 \langle x_f | \overbrace{e^{-iH\Delta t} |x_N\rangle\langle x_N| \cdots |x_1\rangle\langle x_1| e^{-iH\Delta t}}^{\text{Hamiltonian in the given time interval}} |x_i\rangle$$

$$\langle x_i | e^{-iH\Delta t} | x_{i-1}\rangle = \int \frac{dp_i}{2\pi} \langle x_i | p_i\rangle \langle p_i | e^{-iH\Delta t} | x_{i-1}\rangle$$

Assume that V only depends on x

$$\Rightarrow e^{-iV(x_{i-1}, t_{i-1})\Delta t} \int \frac{dp}{2\pi} e^{-i\frac{p_i^2}{2m}} e^{ip(x_i - x_{i-1})}$$

Gaussian integral : $\int dx\, e^{-\frac{1}{2}ax^2 + bx} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}$

$$\Rightarrow \langle x_i | e^{-iH\Delta t} | x_{i-1}\rangle = \sqrt{\frac{2\pi m i}{\Delta t}}\, e^{-i\frac{m}{2\Delta t}(x_i - x_{i-1})}\, e^{-iV(x,t)\Delta t}$$

Let $N \to \infty$ $\quad \overset{N\to\infty}{=} \mathcal{N} e^{iL(x_i,\dot{x}_i)\Delta t}$

$$\therefore \langle x_f | e^{-i(t_f - t_i)H} | x_i\rangle = \mathcal{N} \int dx_N \cdots \int dx_1 \overbrace{e^{iL(x_N,\dot{x}_N)\Delta t} \cdots e^{iL(x_1,\dot{x}_1)\Delta t}}^{e^{i\int dt\, L(x,\dot{x})}}$$

$$= \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{iS[x]}$$

If we consider analytic continuation (Wick rotation)

$t \to i\tau \qquad \tau = -it$

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = -i\frac{d}{d\tau} \quad \Rightarrow \quad \dot{x} = -ix'$$

$$L_E = -\frac{m}{2}x'^2 - V(x) = \text{"}-H\text{"}$$

$$S_E = \int i\, d\tau\, -L_E = -iS_E$$

$$\int \mathcal{D}x\, e^{iS[x]} = \int \mathcal{D}x\, e^{+S_E} \Longleftrightarrow \int_{PB} \mathcal{D}s\, e^{H[s]}$$

* This implies that functional integral is providing a notion for partition funct.
In the quantum statistical mechanics

$$Z = \mathrm{tr}[e^{-\beta H}]$$

$$= \int dx_f \int dx_i\, \delta(x_f - x_i) \langle x_f | \underbrace{e^{H(\beta - 0)}}_{U(\tau_f = \beta,\, \tau_i = 0)} | x_i\rangle$$

$\Rightarrow x_f = x(\beta = \tau), \quad x_i = x(\tau = 0)$

$\Rightarrow$ This condition implies that x obeys the periodic boundary condition with the period $t\beta$.

$$\therefore Z = \int_{PB} \mathcal{D}x\, e^{H[x]} = \int_{PB} \mathcal{D}\eta\, e^{H[\eta]}$$

V.3. Functional Derivative and Correlation Function

For Ising

$$\langle ij \rangle = \frac{\partial}{\partial h_i}\frac{\partial}{\partial h_j} Z \Big|_{h=0} \quad \text{where} \quad Z = \sum_{\{s_i\}} e^{+\beta \Sigma (J s_i s_{i+1} + h_i s_i)}$$

For a quadratic Landau free energy:

$$f_L[\eta] = \vec{\eta}^T M \vec{\eta} + \vec{A}^T \vec{\eta}$$

M : real, symmetric $\langle \eta | M | \eta \rangle + \langle N | \eta \rangle$

A : current vector

i.e. $\eta = (s_1, \cdots, s_N)$

$$M = \begin{pmatrix} 0 & J & \cdot & \cdot \\ J & 0 & J & \cdot\cdot \\ 0 & J & 0 & J \cdot\cdot \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix} \qquad A = (h_1, \cdots, h_N)$$

Generating functional

$$Z[J] = e^{\frac{1}{2} A^T M^{-1} A} \times \frac{(2\pi)^{N/2}}{\sqrt{\det M}} \equiv e^{-W}$$

W: Effective Landau free energy
Generating functional of connected correlation fn.

$$W = -\frac{1}{2} A^T M^{-1} A + \frac{1}{2} \log \det M + \text{const.}$$

$$= -\frac{1}{2} \sum A_i^T (M^{-1})_{ij} A_j + (A\text{-indep. terms})$$

$$\langle \eta_i \eta_j \rangle = \frac{1}{Z} \int_{PB} \mathcal{D}\eta \, \eta_i \eta_j \, e^{H[\eta]}$$

$$\Leftrightarrow \langle \eta_i \eta_j \rangle = \frac{1}{Z} \underbrace{\left(\frac{\delta}{\delta A_i} \frac{\delta}{\delta A_j} \right)}_{\text{functional derivative}} Z \Big|_{\vec{A}=0}$$

$$\frac{\delta f(y)}{\delta f(x)} = \delta(y-x), \quad \frac{\delta}{\delta f(x)} (F_1[f] + F_2[f]) = \frac{\delta}{\delta f(x)} F_1[f] + \frac{\delta}{\delta f(x)} F_2[f]$$

$$\frac{\delta}{\delta f(x)} (F_1[f] F_2[f]) = F_2[f] \frac{\delta}{\delta f(x)} F_1[f] + F_1[f] \frac{\delta}{\delta f(x)} F_2[f]$$

$$\therefore \langle \eta_{i_1} \cdots \eta_{i_n} \rangle = \frac{1}{Z} \left(\frac{\delta}{\delta A_{i_1}} \cdots \frac{\delta}{\delta A_{i_n}} \right) Z \Big|_{\vec{A}=0}$$

※ As we can compute any correlation function by functional derivative of the partition function, correlation function of connected diagram can be obtaind by taking functional derivative to the effective action.

$$\langle \eta_{i} \cdots \eta_{i_n} \rangle_c = \left(\frac{\delta}{\delta A_{i_1}} \cdots \frac{\delta}{\delta A_{i_n}} \right) W[\vec{A}] \Big|_{\vec{A}=0}$$

Only physical information project onto the connectide dia.

i.e.

$$W[A] = \log Z[A] \qquad Z[A] = e^{W[A]}$$

$$\frac{\delta}{\delta A(x)} W[A] = \frac{1}{Z[A]} \frac{\delta Z}{\delta A(x)} = \langle \eta(x) \rangle$$

$$\frac{1}{Z[A]}\frac{\delta}{\delta A(x)}\frac{\delta}{\delta A(y)}Z[A] = \frac{1}{Z[A]}\frac{\delta}{\delta A(x)}\left[\left(\frac{\delta}{\delta A(y)}W[A]\right)e^{W[A]}\right]$$

$$= \frac{1}{Z[A]}\left[\frac{\delta}{\delta A(x)}\frac{\delta}{\delta A(y)}W[A] + \frac{\delta}{\delta A(y)}W[A]\frac{\delta}{\delta A(x)}W[A]\right]e^{W[A]}$$

$$= \langle A(x)A(y)\rangle_c + \langle A(y)\rangle_c\langle A(x)\rangle_c$$

VI. The Renormalisation Group.

VI.1 A Heuristic Approach to Renormalisation

Starting with the Gaussian integration

$$\int dx \int dy \; e^{-\frac{1}{2}(x^2+y^2)} = 2\pi$$

Then how can we integrate with $\epsilon \ll 1$

$$\int dx \int dy \; e^{-\frac{1}{2}(x^2+y^2) - \frac{\epsilon}{2}(x^2 y^2)}$$

$$= \int dx \int dy \; e^{-\frac{1}{2}\cdot 1 x^2} e^{-(\frac{1}{2}+\frac{\epsilon}{2}x^2) y^2}$$

bare

$$= \int dx \; e^{-\frac{1}{2}x^2} \sqrt{\frac{2\pi}{1+\epsilon x^2}} = \sqrt{2\pi}\int dx \; e^{-\frac{1}{2}x^2 - \frac{1}{2}\log(1+\epsilon x^2)}$$

$$= \sqrt{2\pi}\int dx \; e^{-\frac{1}{2}x^2 - \frac{1}{2}(\epsilon x^2 + \mathcal{O}(\epsilon^2))}$$

$$\sim \sqrt{2\pi}\int dx \; e^{-\frac{1}{2}(1+\epsilon)x^2} = 2\pi\left(\frac{1}{\sqrt{1+\epsilon}}\right)$$

Coarse-graining
decimation
renormalisation

renormalised coupling

$$\Rightarrow \int dy \; e^{-\frac{1}{2}\cdot 1 x^2} e^{-\frac{1}{2}(1+\epsilon x^2)y^2} \sim \sqrt{2\pi}\, e^{-\frac{1}{2}(1+\epsilon)x^2}$$

- We obtained "effective action" for x by integrating out y and as a byproduct of the integration, the "coupling constant" for x has been changed or "renormalised".
- As consequence of the renormalisation, the coupling constant is no longer a constant but a function of a parameter ϵ.

VI.2. Kadanoff block spin transformation

Consider the original form of Ising

$$-\beta H(s_i, \{K\}) = \sum_{\{n\}} K^{(n)} \sigma_{i(n)}^{(n)}$$

Block spin transformations : Coarse-graining short wavelength dof (Removing)
(Kadanoff, 1966) and get effective Hamiltonian for long wavelength dof.

$\Rightarrow$ As consequence of the block spin transformation, coupling constants have been changed

$$\vec{r} \longrightarrow \ell\vec{r} \qquad \Rightarrow \qquad [K'] = R_\ell [K] \qquad \ell > 1$$

Does this transformation satisfy the group properties?

1) Closure

$$\vec{r} \rightarrow \ell_1\vec{r} \rightarrow \underbrace{\ell_2\ell_1\vec{r}}_{\ell'\vec{r}}$$

2) Associativity

$$(\ell_1\circ\ell_2)\circ\ell_3 = \ell_1\circ(\ell_2\circ\ell_3)$$

3) Identity

$$\vec{r} \rightarrow 1\cdot\vec{r} = \vec{r}$$

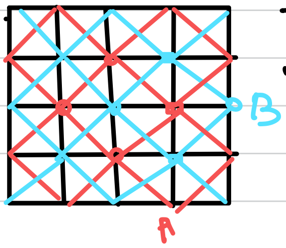
4) Inverse

$$\vec{r} \rightarrow \ell\cdot\vec{r} \rightarrow \ell^{-1}\ell\cdot\vec{r} \stackrel{?}{=} \vec{r}$$

$$\ell^{-1} = \frac{1}{\ell} < 1 \quad : \underline{\text{violate}}$$

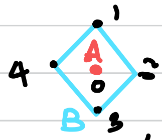
$\Rightarrow$ Block spin transformation does not have inverse elements
$\Rightarrow$ Semi group structure

Example | 2D Ising Model



$$-\beta H \equiv \mathcal{H} = \sum K^{(n)} \sigma^{(n)}_{i(n)}$$

"Integrating out" the lattice A $\Rightarrow$ $l = \sqrt{2}$



$$= \sum_{s_0=\pm 1} e^{J s_0 (s_1+s_2+s_3+s_4)}$$

$$= e^{\frac{1}{2}J'(s_1s_2+s_2s_3+s_3s_4+s_4s_1)} e^{K'(s_1s_3+s_2s_4)} e^{K_4'(s_1s_2s_3s_4)} e^{F'}$$

$\{s_1, s_2, s_3, s_4\} = (+1,+1,+1,+1)$

$$\sum_{s_0=\pm 1} e^{4Js_0} = 2\cosh 4J = e^{2J'+2K'+K_4'+F'}$$

$(1,1,1,-1)$ $\quad 2\cosh 2J = e^{-K_4'+F'} \Rightarrow -K_4' + F' = \log(2\cosh 2J)$

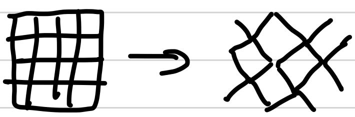
$(1,1,-1,-1)$ $\quad 2 = e^{-2K'+K_4'+F'} \Rightarrow -2K' + K_4' + F' = \log 2$

$(1,-1,1,-1)$ $\quad 2 = e^{-2J'+2K'+K_4'+F'}$

$$\Rightarrow J' = \frac{1}{4}\log\cosh 4J \qquad K_4' = \frac{1}{8}\log\cosh 4J - \frac{1}{2}\log\cosh 2J$$

$$K_2' = \frac{1}{8}\log\cosh 4J \qquad F' = \log 2$$

- Note that we have introduced two new couplings K_2 and K_4 which have not displayed in the original theory.

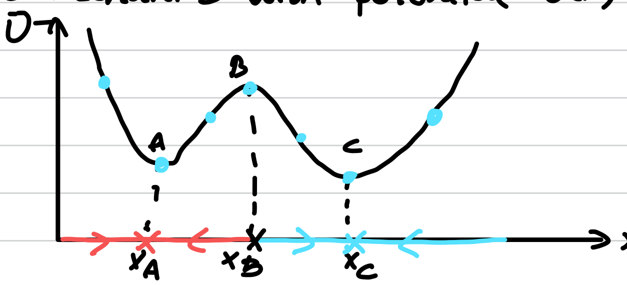


VI.3. Renormalisation and Phase transition

- Fixed point

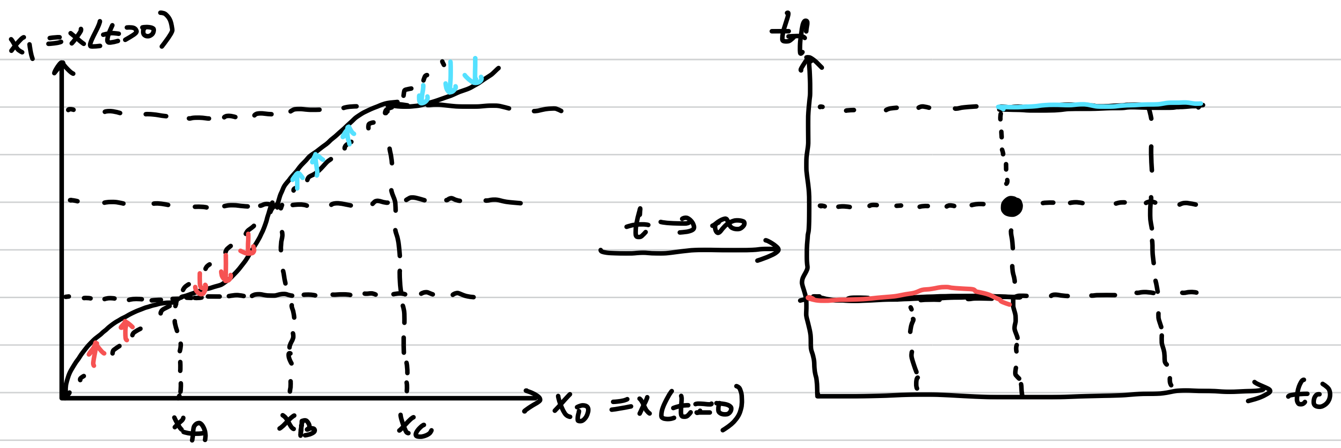
We said $\{K^*\}$ is a fixed point if $R_l[\{K^*\}] = \{K^*\}$, $\forall l$

i.e. 1D mechanics with potential $U(x)$



initial position x_0

① $x_0 < x_A$ $\rightarrow$ reach x_A | Basin of attract. of x_A
② $x_A < x_0 < x_B$ $\nearrow$
③ $x_B < x_0 < x_C$ | Basin of attract. of x_C
④ $x_0 > x_C$ $\rightarrow$ reach x_C
⑤ $x_A, x_B, x_C = x_0$
$\quad\hookrightarrow$ stay x_0 : fixed point.



For 2D Ising, If we ignore K_2' and K_4'

$$J' = \frac{1}{4} \log \cosh 4J$$

On fixed pt, $\quad J'^* = \frac{1}{4} \log \cosh 4J^*$

It seems reasonable result but this equation has no non-trivial solution except $J^* = 0$. One nice way to cure this situation is introducing a new interaction K

$$\tilde{J} \equiv J + K \Rightarrow \tilde{J}^* = \frac{3}{8} \log \cosh 4\tilde{J}^*$$

$\tilde{J}^* = 0.506$ solution

Comparing with Onsager solution $\quad \tilde{J}^* = 0.440$

In chapter II $\quad \tilde{J}^* = \frac{1}{2}\log 3 = 0.549$

From this example, existence of the fixed point is justifying why the new coupling K should be introduced in our renormalised theory.

But why fixed point is significant?

correlation length $\quad \xi[K]$

FP: $[K^*] = R_\ell[K^*]$

$$\left.\begin{aligned} \xi[K^*] &= \frac{1}{\ell}[K^*] \\ &= \xi[K] \end{aligned}\right\} \Rightarrow [K^*] = 0 \text{ or } \infty$$

- For $\xi = 0$ case, it is called the trivial fixed point. It describes a bulk phase.
- For $\xi = \infty$ case, it is called the critical fixed point. It describes the singular critical behaviour which determines critical exponent and universality near critical point.

<u>Note</u> all point in the basin of attraction of a critical fixed point have infinite correlation length.

$$\rightarrow\rightarrow\rightarrow \underset{FP}{\times} \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \; K \qquad \xi[K'] = \frac{1}{\ell}\xi[K]$$

VII. The Renormalisation Group Flow

VII.1. Linearised RG Transformation.

Near fixed point : $K_i = K_i^* + \delta K_i$

$RG \Rightarrow K_i'[K] = K_i^* + \delta K_i'$

Taylor expansion

$$K_i'[K^* + \delta K] = K_i^* + \delta K_j \sum_j \underbrace{\frac{\partial K_i'}{\partial K_j}\bigg|_{K_j = K_j^*}}_{= M_i^{\ j}} + \mathcal{O}\left((\delta K)^2\right)$$

* There is no reason for M_{ij} is either symmetric or diagonalisable but in this lecture we only deal symmetric M_{ij} in most cases.

Under RG transformation $R_\ell[M] = M^{(\ell)}$

diagonalisation : $\sum_j M_i^{\ j} \hat{e}_j^{(a)} = \lambda^{(a)} \hat{e}_i$ a : label for eigenvalue

$$R_\ell[\delta K_i] = \delta K_i' = = M_i^{\ j} \delta K_j$$

$$= M_i^{\ j} \sum_a k^{(a)} \hat{e}_j^{(a)} = \sum_a k^{(a)} \lambda^{(a)} \hat{e}_j^{(a)}$$

$$= \sum_a k'^{(a)} \hat{e}_j^{(a)}$$

As ℓ increases, $k'^{(a)}$ should

1. increase if $|\lambda^a| > 1$ $\Rightarrow$ Relevant
2. decrease if $|\lambda^a| < 1$ $\Rightarrow$ Irrelevant
3. not change if $|\lambda^a| = 1$ $\Rightarrow$ Marginal

$\Rightarrow$ Logarithmic correction

VII.2. RG Flow

- Critical fixed point : A fixed point where the correlation length is diverged.
- Critical manifold : A region in the coupling space where a point flows to the critical fixed point. (i.e. basin of attraction)
- Codimension : Difference between dimensionality of coupling space and critical manifold. $\Rightarrow$ # of irrelevant eigenvectors

Codim = 0 : Sink (bulk phase)

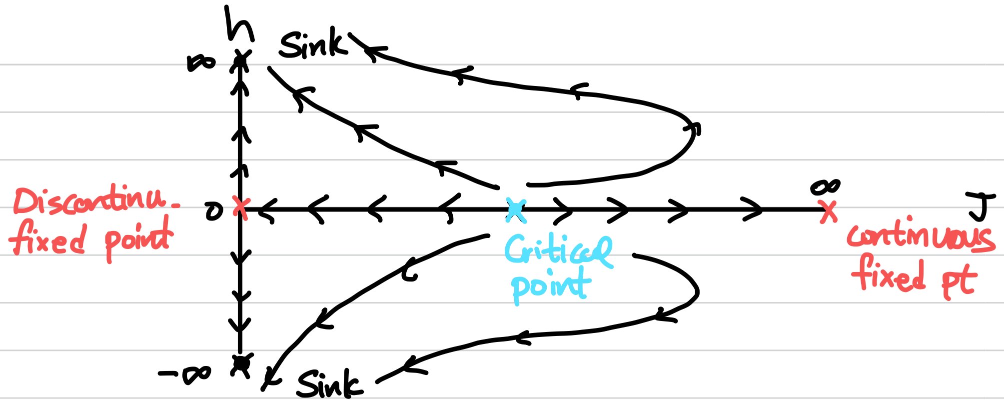
1 : Discontinuous / Continuous FP

$\downarrow$ Plane of coexistence $\searrow$ bulk phase

2 [triple point ($\xi = 0$) ; critical FP ($\xi = \infty$)

>2 [multiple coexistence FP ($\xi = 0$) ; multicritical point ($\xi = \infty$)

i.e. For 2D Ising $[K] = (T, h)$



VII.3. Origin of Scaling Law

- One relevant variable

Consider a system with one relevant variable

$$\longleftarrow\!\!\times\!\!\longrightarrow \quad T$$
T^*

$R_\ell[T] = T'$ $\qquad$ $R_\ell[T^*] = T^*$

Linearised RG $\qquad \delta T = T' - T^*$

$$= R_\ell[T] - R_\ell[T^*]$$

$(t = (T-T^*)/T^*)$ $\qquad = \lambda_\ell (T - T^*) + O((T-T^*)^2)$ $\qquad \lambda_\ell = \left.\frac{dR_\ell}{dT}\right|_{T=T^*}$

$\Rightarrow t' = \lambda_\ell t$

$t^{(1)} \equiv t' = \lambda_\ell t = \ell^{P_t} t \quad \Rightarrow t^{(n)} = \ell^{nP_t} t \qquad \ell^{P_t} > 1,\ P_t > 0$

ⓐ Correlation length

$$\xi^{(1)} = \frac{1}{\ell}\xi(t) = \xi(t')$$

$$\xi(t) = \ell\xi^{(1)} = \ell^n \xi^{(n)} = \ell^n \xi(t^{(n)})$$
$$= \ell^n \xi(\ell^{nP_t} t)$$

Let $\tau \equiv \ell^{nP_t} t$

$$\xi(t) = \ell^n \xi(\tau) = \left(\frac{\tau}{t}\right)^{\frac{1}{P_t}} \xi(\tau) \propto t^{-\frac{1}{P_t}} = t^{-\nu}$$

$$\Rightarrow \nu = \frac{1}{P_t}$$

ⓑ free energy density

$$f(t) = \ell^{-d} f(t^{(1)}) = \ell^{-nd} f(t^{(n)}) = \ell^{-nd} f(\ell^{nP_t} t) = \ell^{-nd} f(\tau)$$

$$\Rightarrow f(t) = \left(\frac{\tau}{t}\right)^{-\frac{d}{P_t}} f(\tau) = \left(\frac{t}{\tau}\right)^{\frac{d}{P_t}} f(\tau)$$

$$C_v \propto \frac{\partial^2 f}{\partial t^2} \propto t^{\frac{d}{P_t}-2} = |t|^{-\alpha} \quad \Rightarrow \alpha = 2 - \frac{d}{P_t} = 2 - \nu d$$

- Two relevant variables

$$T' = R_\ell^T (T, h)$$
$$h' = R_\ell^h (T, h)$$

Linearised RG $\begin{pmatrix} \delta T' \\ \delta h' \end{pmatrix} = M \begin{pmatrix} \delta T \\ \delta h \end{pmatrix}$ evaluate on $T=T^*$, $h=h^*$

$$= \begin{pmatrix} \partial_T R_\ell^T & \partial_h R_\ell^T \\ \partial_T R_\ell^h & \partial_h R_\ell^h \end{pmatrix} \begin{pmatrix} \delta T \\ \delta h \end{pmatrix}$$

Suppose that $\begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} \lambda_\ell^t & 0 \\ 0 & \lambda_\ell^h \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix}$

$$\lambda_\ell^t = \left.\frac{\partial R_\ell^T}{\partial t}\right|_{(t,h)=(t^*,h^*)} \qquad \lambda_\ell^h = \left.\frac{\partial R_\ell^h}{\partial h}\right|_{(t,h)=(t^*,h^*)}$$

ⓐ Correlation length

$$\xi(t,h) = \ell\,\xi(t',h') = \ell^n \xi(\ell^{np_t} t, \ell^{np_h} h)$$

for $h=0$ $\quad \ell^{np_t} = \tau \Rightarrow \xi \sim t^{-\nu}, \quad \nu = 1/p_t$

for $t=0$ $\quad \xi(h') = \ell^n \xi(\ell^{np_h} h)$

$$= \left(\frac{h}{\rho}\right)^{-\frac{1}{p_h}} \xi(\rho) \sim (h)^{-\frac{1}{p_h}}$$

ⓑ free energy

$$f(t,h) = \ell^{-d} f(t^{(1)}, h^{(1)}) = \ell^{-nd} f(t^{(n)}, h^{(n)}) = \ell^{-nd} f(\tau, \ell^{np_h} h)$$

$$= \ell^{-nd} f(\ell^{np_t} t, \ell^{np_h} h)$$

$\ell^n = \left(\frac{\rho}{h}\right)^{\frac{1}{p_h}}$

$$= \left(\frac{\rho}{h}\right)^{-\frac{d}{p_h}} f\left(\left(\frac{\rho}{h}\right)^{\frac{p_t}{p_h}}, \rho\right)$$

$$m \propto \frac{\partial f}{\partial h} \propto h^{\frac{p_h}{d}-1} = h^{1/\delta} \quad \Rightarrow \delta = \frac{d-p_h}{p_h}$$

Otherwise

$$f(t,h) = \left(\frac{t}{\tau}\right)^{\frac{d}{p_t}} f\left(\tau, \left(\frac{t}{h}\right)^{-p_h/p_t} h\right)$$

set $\quad w \equiv p_t/d, \quad u = p_h/d$

$f(t,h) = |t|^{\frac{1}{w}} \Phi\left(\frac{h}{|t|^{u/w}}\right)$ coincides with the scaling hypothesis.

- With irrelevant variables

If there are irrelevant variables $K, K', \cdots$

$$f(t, \cdots, \underbrace{K, K', \cdots}_{\text{irrelevant}}) = \ell^{-d} f(t\lambda^t, \cdots, K\lambda^k, K'\lambda^{k'}, \cdots)$$

$p_k, p_{k'}, \cdots < 0$

$$= \ell^{-nd} f(t\ell^{np_t}, \cdots, \underbrace{K\ell^{np_k}, K'\ell^{np_{k'}}}, \cdots)$$

they might be vanished as $n \to \infty$

$$= \left(\frac{\tau}{t}\right)^{d/p_t} f(\tau, \cdots, 0, 0, \cdots)$$

As we renormalise free energy, the irrelevant variables become ignorable.
However, if $f \propto 1/K$, for example, it seems not to ignore.
We called this kind of variables as Dangerously irrelevant variable.
i.e. $f = at\phi^2 + \frac{1}{2}u\phi^4$

$$\frac{\partial f}{\partial \phi} = 2at\phi + 2u\phi^3 = 2\phi[at + u\phi^2] = 0$$

$$\Rightarrow \phi^2 = -\frac{a}{u}t$$

$$f = at\left(-\frac{at}{u}\right) + \frac{1}{2}u\left(\frac{at}{u}\right)^2 = -\frac{1}{2}\frac{(at)^2}{u}$$

Even though u is irrelavant variable, f diverges as $u \to 0$
so u is a dangerously irrelevant variable.

VIII. Gaussian Model and Beyond

VIII.1. Gaussian Model as a Free Theory

Previously, we have considered a Landau theory with quartic term

$$f_L[\phi(x)] = a\phi^2 + b\phi^4$$

Here, we only consider upto quadratic term with gradient

$$\mathcal{H}[\phi(x)] = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0(t)\,\phi^2 \longrightarrow \text{"mass"}$$

$\hookrightarrow \vec{\nabla} = \hat{e}_i \partial^i$: fluctuation along $\hat{e}_i$ direction

$$S_0 = \int_a d^dx\, \mathcal{H}[\phi(x)] = \int_a d^dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2\right]$$

$$= \int_a d^dx \left[\frac{1}{2}\phi(x)\left(-\nabla^2 + r_0(t)\right)\phi(x)\right]$$

$$= \int_a d^dx\, d^dy \left[\frac{1}{2}\phi(x)\underbrace{\left(-\nabla^2 + r_0(t)\right)\delta^d(x-y)}_{\substack{\shortparallel \\ G^{-1}(x-y)}}\phi(y)\right]$$

$$G(x-y) = \frac{\delta^d(x-y)}{\text{"}-\nabla^2 + r_0(t)\text{"}} = \int^{\Lambda}\frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{k^2 + r_0(t)}$$

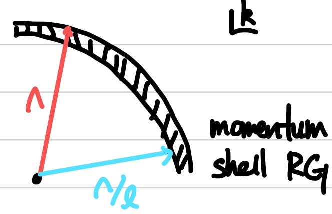
$$\propto \frac{1}{r^{d-2}} \implies \text{No anomalous dimension}$$

Decimation of small distance $\Rightarrow$ integrating out high momentum

$\phi \rightsquigarrow 0 < |k| < \Lambda$

$$\hookrightarrow \begin{cases} \phi_S^\ell(k) & \text{for } 0 < |k| < \frac{\Lambda}{\ell} \\ \phi_F(k) & \text{for } \frac{\Lambda}{\ell} < |k| < \Lambda \end{cases}$$

ϕ_F: short wavelength, large momentum, fast mode



$$S_0 = S_S + S_F = \int_0^{\Lambda/\ell}\frac{d^dk}{(2\pi)^d}\phi_S(-k)(k^2 + r_0)\phi_S(k) + \int_{\Lambda/\ell}^{\Lambda}\frac{d^dk}{(2\pi)^d}\phi_F(-k)(k^2 + r)\phi_F(k)$$

$$Z = \int \mathcal{D}\phi_S^\ell\, \mathcal{D}\phi_F\, e^{-(S_S + S_F)}$$

$$= \int \mathcal{D}\phi_S^\ell\, e^{-S_S[\phi_S]}\int \mathcal{D}\phi_F\, e^{-S_F[\phi_F]}$$

$$= \int \mathcal{D}\phi_S^\ell\, e^{-S_S[\phi_S]}Z_F = \int \mathcal{D}\phi_S^\ell\, e^{-\underbrace{(S_S[\phi_S] - \log Z_F)}}$$

effective action $S[\phi_S^\ell]$

$$\Rightarrow \begin{cases} \text{Field renormalisation} & \phi'(k) = z_s\phi_S^\ell(k_\ell) \\ \text{Momentum rescaling} & k_\ell = \ell k \end{cases}$$

But for Gaussian, Z_F is a regular function on r_0 so we can ignore it.

$$Z = \int \mathcal{D}\phi_S^\ell\, e^{-S_S[\phi_S^\ell]}$$

$$= \int \mathcal{D}\phi'\, e^{-\int_0^\Lambda \frac{d^dk_\ell}{(2\pi)^d}\ell^{-d}\frac{1}{2}(r_0 + \ell^{-2}k_\ell^2)\phi'(k_\ell)\phi'(-k_\ell)z_s^2}$$

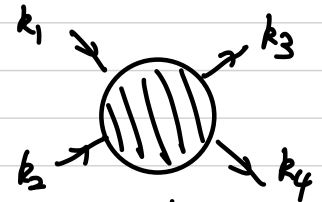
$$= \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_0^\Lambda \frac{d^d k_\ell}{(2\pi)^d} \ell^{-d-2} (k_\ell^2 + \underbrace{\ell^2 r_0}_{= R_\ell[r_0]}) Z_s^2 |\phi'|^2}$$

- $R_\ell[r_0] = r_0' = r_0 \Rightarrow r_0 = 0$: Gaussian fixed point
- Since $r=0$ is a fixed point, the theory should be invariant

$$\frac{1}{2}\ell^{d-2} k_\ell^2 Z_s^2 |\phi'|^2 \overset{!}{=} \frac{1}{2}k^2|\phi|^2 \Rightarrow Z_s = \ell^{\frac{d}{2}+1}$$

- $r_\ell = \ell^2 r \Rightarrow r$ is a relevant coupling.

VIII.2. ϕ^4 Interaction and ϵ expansion



$$S = \int_a d^dx \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + \frac{1}{4!}u_0\phi^4\right)$$

$$\equiv S_0 + V \qquad V = \int_a d^dx \, \frac{u_0}{4!}\phi^4$$

momentum conservation

$$= \int^\Lambda \frac{d^dk}{(2\pi)^d}\left(\phi(-k)(k^2+r_0)\phi(k)\right) \int^\Lambda \prod_{i=1}^4 \frac{d^dk_i}{(2\pi)^d} u_0 \left(\delta^d(\textstyle\sum k)\, \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\right)$$

$$S[\phi_s, \phi_F] = S_s[\phi_s] + S_F[\phi_F] + V[\phi_s,\phi_F]$$

$$Z = \int \mathcal{D}\phi_s \, e^{-S_s} \int \mathcal{D}\phi_F \, e^{-S_F - V}$$

$$e^{-S_{eff}[\phi_s]} = e^{-S_s} \int \mathcal{D}\phi_F \, e^{-S_F - V}$$

$$= e^{-S_s[\phi_s]} \frac{\int \mathcal{D}\phi_F \, e^{-S_F[\phi_F] - V[\phi_s,\phi_F]}}{\int \mathcal{D}\phi_F \, e^{-S_F[\phi_F]}} \cancel{\int \mathcal{D}_F \, e^{-S_F[\phi_F]}}^{\text{const.}}$$

$$= e^{-S_s[\phi_s]} \langle e^{-V} \rangle_F$$

$$= e^{-S_s[\phi_s]} + e^{-\langle V\rangle_F + \frac{1}{2}(\langle V^2\rangle_F - \langle V\rangle_F^2)} + \cdots$$

※ Cumulant expansion $\log\langle e^V\rangle_F = \log[1 + \langle V\rangle_F + \frac{1}{2}\langle V^2\rangle_F + \cdots] = \langle V\rangle_F + \frac{1}{2}[\langle\ \rangle - \langle\ \rangle] + \cdots$

$$\Rightarrow S_{eff}[\phi_s] = S_s + \langle V\rangle_F - \frac{1}{2}(\langle V^2\rangle_F - \langle V\rangle_F^2) + \cdots$$

To wrap up this effective action, we can use Feynman diagram.

$$\langle V\rangle_F = u_0 \frac{1}{2!}\frac{1}{2!} \left\langle \int^\Lambda \prod_{i=1}^4 \frac{d^dk_i}{(2\pi)^d} \underbrace{(\phi_F+\phi_s)(k_1)(\phi_F+\phi_s)(k_2)(\phi_F+\phi_s)(k_3)(\phi_F+\phi_s)(k_4)\delta^d}_{\cancel{\phi_F^4},\ \cancel{\phi_F^3\phi_s}\quad \phi_F^2\phi_s^2 \quad \cancel{\phi_F\phi_s^3} \quad \phi_s^4} \right\rangle$$

$\cancel{\phi_F^4}$: constant contribution

$\cancel{\phi_F^3\phi_s}$, $\cancel{\phi_F\phi_s^3}$ $\Leftarrow$ symmetry $\phi \to -\phi$

$\phi_F^2\phi_s^2$: one-loop contribution

ϕ_s^4: tree level

$$S_{tree} = \frac{u_0}{4!}\int^{\Lambda/\ell} \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d}\, \delta^d(\Sigma k_i)\, \phi_s(k_4)\,\phi_s(k_3)\,\phi_s(k_2)\,\phi_s(k_1)$$

$k_\ell = \ell k$

$\phi'(k) = Z_s \phi_s^\ell(k_\ell)$

$$= \frac{u_0}{4!}\int_0^\Lambda \prod_{i=1}^{4} \ell^{-4d} \prod_{i=1}^{4} \frac{d^d (k_\ell)_i}{(2\pi)^d}\, \ell^d\, \delta(\Sigma k_\ell)\, Z_\phi^4\, \phi_s^4$$

$$= \underbrace{\frac{1}{4!} u_0 \ell^{-3d} Z_\phi^4}_{} \int_0^\Lambda \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d}\, \delta(\Sigma k)\, \phi_s^4$$

$$u_\ell = u_0 \ell^{-3d}\, \ell^{4+2d} = u_0 \ell^{4-d}$$

$d<4$: relevant

$d>4$: irrelevant

$$S_{loop} = \frac{3u_0}{4!}\int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d}\, \delta(\Sigma k)\, \phi_s(k_4)\,\phi_s(k_2)\langle \phi_F(k_3)\,\phi_F(k_1)\rangle$$

$k_1+k_2-k_3-k_4$

$$= \frac{3u_0}{4!}\int^{\Lambda/\ell} \frac{d^d k_2}{(2\pi)^d}\frac{d^d k_4}{(2\pi)^d}\, \phi_s(k_2)\,\phi_s(k_4)\left(\int_{\Lambda/\ell}^{\Lambda} \frac{d^d k_3}{(2\pi)^d}\frac{d^d k_1}{(2\pi)^d}\, \frac{\delta^d(k_3-k_1)}{k_3^2+r_0}\,(2\pi)^d\right)\delta(\Sigma k)$$

$$= \frac{3u_0}{4!}\int_0^{\Lambda/\ell} \frac{d^d k}{(2\pi)^d}\, \phi_s(k)\,\phi_s(k)\left(\int_{\Lambda/\ell}^{\Lambda} \frac{d^d k'}{(2\pi)^d}\, \frac{1}{(k')^2+r_0}\right)$$

$$= \frac{3u_0}{4!}\ell^{-d} Z_s^2 \int_0^\Lambda \frac{d^d k_\ell}{(2\pi)^d}\left(\phi_s^\ell(k_\ell)\right)^2 \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k'}{(2\pi)^d}\, \frac{1}{(k')^2+r_0}$$

$$\Rightarrow S_{eff} = Z_\phi^2 \ell^{-d-2} \int_0^\Lambda \frac{d^d k_\ell}{(2\pi)^d}\left[k_\ell^2 + r_0\ell^2 + \frac{3}{4!}u_0\ell^2 \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k'}{(2\pi)^d}\frac{1}{(k')^2+r_0}\right]\left(\phi_s^\ell(k_\ell)\right)^2$$

$$+ \frac{1}{4!} u_0 \ell^{-3} Z_\phi^4 \int_0^\Lambda \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d}\, \delta(\Sigma k)\, \phi_s^k$$

$Z_\phi = \ell^{d/2+1}$

ℓ^{4-d} $\rightsquigarrow$ expand near $d=4$ (or $d=4-\epsilon$)

$$r_\ell = r_0\ell^2 + 3u_0\ell^2 \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+r_0}$$

$$\int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+r_0} = \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d}\frac{1}{k^2}\left(1-\frac{r_0}{k^2}+\mathcal{O}(r_0^2)\right) \qquad \Lambda \gg r_0$$

$$= \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d}\left(\frac{1}{k^2}-\frac{r_0}{k^4}\right) \qquad \int \frac{d^4 k}{(2\pi)^4} = \int d|k|\, \frac{2\pi^2}{(2\pi)^4} k^3$$

$$= \int_{\Lambda/\ell}^{\Lambda} \frac{d|k|}{8\pi^2}\left(k - r_0\frac{1}{k}\right) = \frac{1}{8\pi^2}\left[\frac{\Lambda^2}{2}\left(1-\frac{1}{\ell^2}\right) - r_0 \log \ell\right]$$

$$\Rightarrow r_\ell = \ell^2\left[r_0 + \frac{3}{4\pi^2}\left(\frac{u_0}{4!}\right)\Lambda^2\left(1-\frac{1}{\ell^2}\right) - \frac{3}{2\pi^2}\frac{u_0}{4!} r_0 \log\ell + \mathcal{O}(r_0^2)\right]$$

$$u_\ell = u_0\ell^\epsilon = u_0 e^{\epsilon\log\ell} \simeq 1+\epsilon\log\ell$$

* Beta function

$$\beta_r = \frac{\partial r_\ell}{\partial(\log\ell)} = \ell\frac{\partial r_\ell}{\partial\ell} \qquad \beta_u = \frac{\partial u_\ell}{\partial(\log\ell)} = \ell\frac{\partial u_\ell}{\partial\ell}$$

$\beta = 0 \Rightarrow$ fixed point.

$$r_{\ell+\delta_\ell} = r_\ell + \delta r = (\ell+2\delta\ell)\left[r_0 + \frac{3}{4\pi^2}\left(\frac{u_0}{4}\right)\Lambda^2\left(1-\frac{1}{\ell^2}\left(1+\frac{\delta\ell}{\ell}\right)^2\right) - \frac{3}{2\pi^2}\frac{u_0}{4} r_0 \log\ell\left(1+\frac{\delta\ell}{\ell}\right)\right]$$

$$= r_\ell + 2r_\ell\frac{1}{\ell}\delta_\ell + \frac{3}{2\pi^2}\frac{u_0}{4}\Lambda^2\frac{1}{\ell}\delta\ell - \frac{3}{2\pi^2}\frac{u_0 r_0}{4}\ell\delta\ell$$

$$\Rightarrow\ l\frac{\partial r_l}{\partial l} = 2r_l + \frac{3}{2\pi^2}\frac{u_0}{4}\Lambda^2 - \frac{3}{2\pi^2}\frac{u_0}{4}\underbrace{r_0 l^2}_{r_l+\mathcal{O}(\epsilon)} = \beta_r(l)$$

$$\beta_u(l) = \epsilon$$

FP : Gaussian FP $(r_l^*, u_l^*) = (0,0)$

$\Rightarrow$ Since we didn't finished 1-loop analysis so we obtain only for the Gaussian fixed point.

VIII.3. Feynman Diagram : Prelude

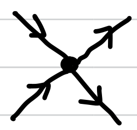
——— propagator (leg) $\bullet$ vertex (A point where momentum conservation is satisfied)

- For ϕ^4-interaction, four propagators meet at a vertex.
- Momentum conservation must be satisfied at all vertices.
- Each external leg is consist of slow mode.

$\longrightarrow\ = \langle \phi_s \phi_s \rangle$ $\quad\cdots\cdots = \langle \phi_F \phi_F \rangle$

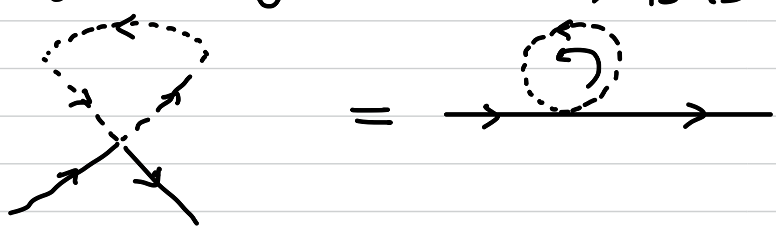
- Disconnected diagrams or diagrams without one or no external leg do not contribute to the effective action

① $\langle \phi_s \cdots \rangle = \int \prod \frac{d^dk}{(2\pi)^d}\ \phi_s \phi_s \phi_s \phi_s\, \delta^d(\Sigma)$

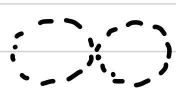


We call a diagram without any loop as tree diagram.

② $\langle \phi_s\phi_s\phi_F\phi_F \rangle = \int^{\Lambda/l} dk\, dk \int^{\Lambda} dk\, dk\ \delta(\Sigma k)\ \phi_s \phi_s \underbrace{\langle \phi_F \phi_F \rangle}_{\delta(k+k)}$

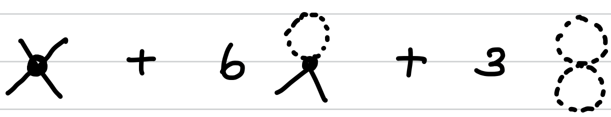


③ $\langle \phi_F \phi_F \phi_F \phi_F \rangle = \langle \phi_F \phi_F \rangle \langle \phi_F \phi_F \rangle$; Wick theorem



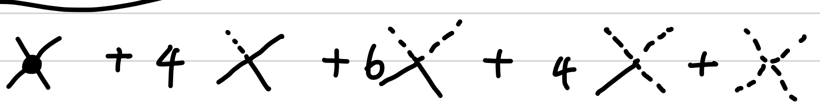
; bubble diagram

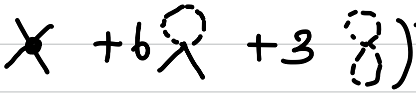
④ $\langle \phi_F \phi_F \phi_F \phi_s \rangle$

$\langle V \rangle_F =$ 

VIII. 4. Next Cumulant and 1-loop Correction

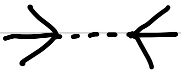
$\langle V^2\rangle_0 - \langle V\rangle_0^2 \longrightarrow (\phi\ \phi\ \phi\ \phi)(\phi\ \phi\ \phi\ \phi)$

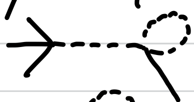
$\langle V^2\rangle_F = ($  $)^2$

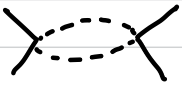
$\langle V\rangle_F^2 = ($  $)^2$

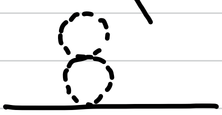
Joint two diagrams while
- Connect with proper propogator
- Make loops with fast mode
- Make external legs with slow mode

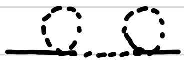
i)  ⇒ Disconnected ⇒ Removed by $\langle V\rangle_F^2$

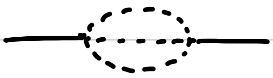
ii) 16 

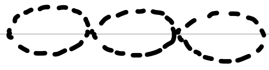
iii) 96 

iv) 72 

v) 144 

vi) 144 

vii) 96 

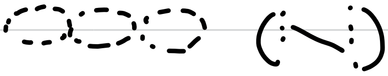
viii) 72 

ix) 24 

※ Geometric factor

iv 

$\frac{6}{3} \times \frac{6}{3} \times 2 = 72$

viii) 

$6 \times 2 \times 6 = 72$

iii) 

three

$4 \times 4 \times 3 = 96$

Since diagram iii) and vi) violates momentum conservation and diagram viii) and ix) are just corresponding to the energy shift so they do not affect on the effective action.

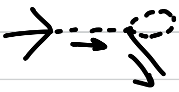
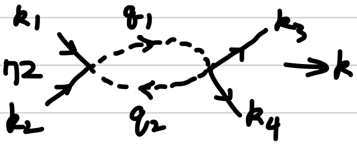


Diagram v) and vii) contribute to renormalisation of r_0 as two-loop contribution, diagram iv) contribute to running of u as one-loop contribution, and diagram ii) contribute to ϕ^6 coupling. (i.e. $g_\ell \neq 0$)
Here, we only consider ϕ^4-interaction up to one-loop so let us consider

$k = k_1 + k_2 = k_3 + k_4 \qquad \Sigma k = k_1 + k_2 - k_3 - k_4 = 0$



$$= \int_0^{\Lambda/\ell} \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d}\, \phi_s(k_1)\phi_s(k_2)\phi_s(k_3)\phi_s(k_4)\, \delta^d(\Sigma k)$$

$$\times \left(-\frac{u_0}{4!}\right)^2 \times 72\, (2\pi)^{2d}\, \underbrace{\delta^d(k_1+k_2+q_1-q_2)\, \delta^d(q_1+k_3+k_4-q_2)}$$

$$\times \int_{\Lambda/\ell}^{\Lambda} \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 + r_0} \frac{1}{q_2^2 + r_0}$$

$$= \frac{u_0^2}{8} \int_0^{\Lambda/\ell} \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{d^d k_4}{(2\pi)^d} (2\pi)^d\, \phi_s(k_1)\phi_s(k_2)\phi_s(k_3)\phi_s(k_4)\, \delta^d(\Sigma k)$$

$$\times \int_{\Lambda/\ell}^{\Lambda} \frac{d^d q_1}{(2\pi)^d} \frac{1}{(q_1+k_1+k_2)^2 + r_0} \frac{1}{q_1^2 + r_0}$$

$k_\ell = \ell k_1$

$$u_\ell = u_0 \ell^\epsilon \left[1 - \frac{u_0}{8}\int_{-\Lambda/\ell}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(r_0+q^2)^2}\right]$$

$$\int_{\Lambda/\ell}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(r+q^2)^2} \simeq \int_{\Lambda/\ell}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{q^4} + \mathcal{O}(r_0^2)$$

$$= \frac{1}{8\pi^2}\int_{\Lambda/\ell}^{\Lambda} dq \, \frac{1}{q} = \frac{1}{8\pi^2} \log \ell$$

$$\therefore u_\ell = u_0 \ell^\epsilon \left[1 - \frac{3}{8\pi^2}\frac{u_0}{4!} \log \ell\right] + \mathcal{O}(u_0^2 r_0)$$

$$\simeq u_0 \left[1 + \left\{\epsilon - \frac{3}{8\pi^2}\frac{u_0}{4}\right\} \log \ell\right]$$

$$\beta_u = \frac{u_0}{4}\left[\epsilon - \frac{3}{8\pi^2}\frac{u_0}{4}\right] \simeq \frac{u_\ell}{4}\left[\epsilon - \frac{3}{8\pi^2}\frac{u_\ell}{4}\right] + \mathcal{O}(\epsilon^2)$$

$$\beta_u = 0 \Rightarrow u_\ell = 0 \text{ or } \frac{8\pi^2}{3}\epsilon$$

$$\beta_r = 2r_\ell + \frac{3}{2\pi^2}\frac{u_0}{4}\Lambda^2 - \frac{3}{2\pi^2}\frac{u_0}{4} r_\ell \ell^2 \overset{!}{=} 0$$

$$(r^*, u^*) = \underset{\text{GFP}}{(0,0)} \quad \text{or} \quad \underset{\text{Wilson-Fischer Fixed Point}}{\left(-\frac{\Lambda^2}{6}\epsilon, \frac{8\pi^2}{3}\epsilon\right)}$$

VII. 5. Wilson-Fischer Fixed Point

- Linearised RG near Gaussian FP

$$M = \left.\begin{pmatrix} \frac{\partial r_\ell}{\partial r_0} & \frac{\partial r_\ell}{\partial u_0} \\ \frac{\partial u_\ell}{\partial r_0} & \frac{\partial u_\ell}{\partial u_0} \end{pmatrix}\right|_{(r_0,u_0)=(0,0)}$$

$$\left.\frac{\partial r_\ell}{\partial r_0}\right| = \ell^2 - \frac{3}{2\pi^2}\frac{u_0}{4!}\log \ell \Big| = \ell^2$$

$$\left.\frac{\partial r_\ell}{\partial u_0}\right| = \frac{1}{4!}\left[\frac{3}{4\pi^2}\Lambda^2\left(1-\frac{1}{\ell^2}\right) - \frac{3}{2\pi^2} r_0 \log \ell\right]\Big| = \frac{1}{4!}\frac{3}{4\pi^2}\Lambda^2(\ell^2-1)$$

$$\left.\frac{\partial u_\ell}{\partial r_0}\right| = 0 \qquad \left.\frac{\partial u_\ell}{\partial u_0}\right| = \ell^\epsilon\left[1 - \frac{3}{8\pi^2}\frac{u_0}{4!}\log \ell\right] - \frac{3}{8\pi^2}\frac{u_0}{4!}\ell^\epsilon \log \ell \Big|$$

$$= \ell^\epsilon$$

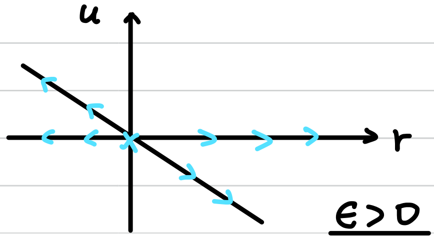
$$M = \begin{pmatrix} \ell^2 & \frac{1}{4!}\frac{3}{4\pi^2}\Lambda^2(\ell^2-1) \\ 0 & \ell^\epsilon \end{pmatrix} \qquad \underbrace{\lambda^{(r)} = \ell^2}_{\text{relavant}} \qquad \underbrace{\lambda^{(2)} = \ell^\epsilon}$$

$\epsilon > 0$: relavant $\Leftrightarrow d < 4$

$\epsilon < 0$: irrelavant $\Leftrightarrow d > 4$

$$\lambda^{(r)} = \ell^2 \Rightarrow e^{(r)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda^{(2)} = \ell^\epsilon \Rightarrow e^{(2)} = \begin{pmatrix} -\frac{1}{4!}\frac{3}{4\pi^2}\Lambda^2(\ell^2-1) \\ 1 \end{pmatrix}$$



$\epsilon > 0$

- Linearised RG near Wilson–Fischer fixed point

$$(r^*, u^*) = \left(-\frac{\Lambda^2}{6}\epsilon, \frac{8\pi^2}{3}\epsilon\right)$$

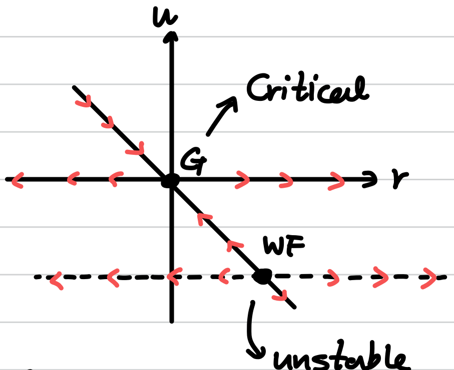
$$\left.\frac{\partial r_\ell}{\partial r_0}\right| = \ell^2\left[1 - \frac{\epsilon}{3}\log \ell\right] \simeq \ell^2 e^{-\frac{\epsilon}{3}\log \ell} = \ell^{2-\frac{\epsilon}{3}}$$

$$\left.\frac{\partial r_\ell}{\partial u_0}\right| = \frac{1}{4!}\frac{3}{4\pi^2}\Lambda^2(\ell^2-1) + \frac{1}{4\pi^2}\Lambda^2\epsilon\log \ell$$

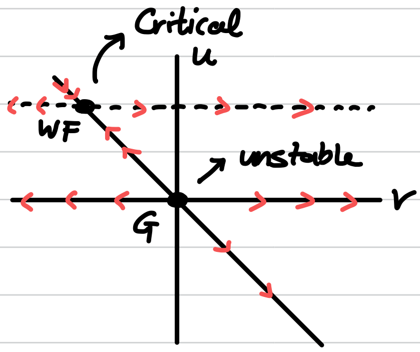
$$\left.\frac{\partial r_\ell}{\partial r_0}\right| = 0 \qquad \left.\frac{\partial u_\ell}{\partial u_0}\right| = 1 - \epsilon\log \ell \simeq \ell^{-\epsilon}$$

$$M = \begin{pmatrix} \ell^{2-\frac{\epsilon}{3}} & \cdots \\ 0 & \ell^{-\epsilon} \end{pmatrix} \qquad \lambda^{(1)} = \ell^{2-\frac{\epsilon}{3}} \qquad \lambda^{(2)} = \ell^{-\epsilon}$$

$\epsilon < 0$



$\epsilon > 0$



Non-physical WF $\Rightarrow u < 0$

IX. Symmetries and Spontaneous Breaking

IX. 1. (Continuous) Symmetry and (Lie) Group / Algebra

<u>Definition</u>) Group $(G, \cdot)$

Group $(G, \bullet)$ is a duet of a set G and an operation $\bullet$ and satisfying three properties

- Closure under $\bullet$: $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$
- Identity and Inversion : For $e \in G$, $\forall g \in G : g \cdot e = e \cdot g = g$
  $\exists g^{-1}$ for $g \in G$: $g^{-1} g = g \cdot g^{-1} = e$
- Associativity : $g_1, g_2, g_3 \in G$, $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

<u>Example</u>)

- $(\mathbb{R}, \bullet)$ is a group.
- $(\mathbb{R}/\{0\}, \times)$ is a group.
- A vector space should be a group for addition.
- $\{e^{2\pi i/n} \mid n \in \mathbb{N}\}$ is a group with respect to the multiplication.

<u>Definition</u>) Order and Continuous/Lie Group

Order is the number of the group element.
If a group has the infinite number of element and if these elements can be parametrised continuously, then we call it as a continuous group.
When this function is smooth, it is called Lie group.

* Note that we will focus on the Lie group in this lecture since Lie group is regarded as a fundamental description of the nature.
* It is convenient to consider group is a duet of a matrix set and matrix multiplication. We call it as the matrix representation.

<u>Example</u>)

- $\{e^{2\pi i/n} \mid n \in \mathbb{N}\}$ is a finite group with order n.
- $\{e^{i\alpha} \mid \alpha \in \mathbb{R}\}$ is a Lie group.
- $\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, \det g = 1\}$ is also a Lie group.

* If all elements in a group is commute each other (e.g. $g_1 \cdot g_2 = g_2 \cdot g_1$, $\forall g_1, g_2 \in G$) then this group is called abelian. If not, it is called non-abelian.

i.e. $\{e^{i\alpha}\}$ is an abelian but $\{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$ is non-abelian.

<u>Definition</u>) Lie algebra/bracket and Generator

Lie algebra is an "vector space near identity" and it is denoted as $\mathfrak{g}$.
Multiplication between Lie algebra is defined as Lie bracket $[\cdot,\cdot]$ satisfy

- Anti-symmetry $[a, b] = -[b, a]$, $a, b \in \mathfrak{g}$
- Jacobi identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Basis of Lie algebra is called generator. So the dimensionality of a Lie algebra is same as the number of generators.
Generator is closed under Lie bracket $[J_a, J_b] = \sum_c f_{ab}{}^c J_c$

<u>Example</u>) 3d rotation.

$$R_x(\theta) = \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos\theta & & +\sin\theta \\ & 1 & \\ -\sin\theta & & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$$

For $R_x(\theta)$

$$R_x(\epsilon) = \begin{pmatrix} 1 & & \\ & 1 & -\epsilon \\ & \epsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \epsilon \underbrace{\begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}}_{L_x \text{ : generator}}$$

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[L_x, L_y] = L_x L_y - L_y L_x = L_z \implies [L_i, L_j] = \epsilon_{ijk} L_k$$

$$\mathfrak{g} = \mathrm{Span}\{L_x, L_y, L_z\}, \quad [L_i, L_j] = \epsilon_{ijk} L_k \quad : so(3)$$

Here, Lie group is presented by upper case, on the other hand, Lie algebra lowercase.

* **Classification**

S : special : $\det = 1$

O : orthogonal $O^T \mathbb{1} O = \mathbb{1}$

U : unitary $U^\dagger \mathbb{1} U = \mathbb{1}$

L : Linear

i.e

$GL(n, \mathbb{C})$: $n\times n$ matrices with invertable and whose entries are complex.

$SL(n, \mathbb{C})$: $n\times n$ matrices with unit determinant.

$O(n)$ $SO(n)$ $U(n)$ $SU(n)$

Dimensionality : $gl(n, \mathbb{R}) : n^2$ $\quad sl(n, \mathbb{R}) : n^2 - 1$

$o(n) \simeq so(n) : \frac{n(n-1)}{2}$ $\quad u(n) = n^2$ $\quad su(n) = n^2 - 1$

IX.2. O(n) Model

Consider N-order parameters ϕ_i $(i = 1, \cdots, N)$

$$\sum_{i=1}^{N} \frac{1}{2} \phi_i (-\nabla^2 + r^2) \phi_i = \frac{1}{2} \vec{\phi}^T M \vec{\phi}$$

$$\vec{\phi}^T = (\phi_1\ \phi_2\ \cdots\ \phi_N)$$

$$M = \mathrm{diag}(-\nabla^2 + r^2)$$

Let us impose a constraint $\vec{\phi}^T \vec{\phi} = \vec{\phi}^2 = 1$

$\Rightarrow O(N)$ constraint

But if we choose a vector $\hat{a} = (1, 0, 0, \cdots, 0)$

$$\phi(x) = D(g)\, \hat{a}$$

↳ Matrix represent. of $O(N)$ element g

and $\hat{a}$ is invariant for $g \in O(N-1) \Rightarrow O(N-1)$: Little group.

* Since $O(N)$ has a little group $O(N-1)$, we can divide $o(N)$ generators into $o(N-1)$ part and complementary ones.

<u>Example</u> $O(1) \simeq \mathbb{Z}_2$: Ising
$O(2)$: XY model
$O(3)$: Heisenberg model ..

Parameterisation : $\vec{\phi}(x) = (\sigma(x), \vec{\pi}^T(x))$
$\hookrightarrow \sigma(x) = \sqrt{1-\vec{\pi}^2(x)}$

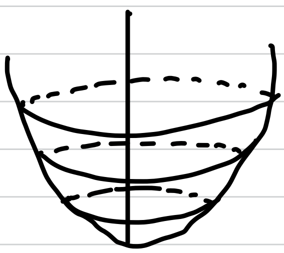
$$\mathcal{H} = \frac{1}{2}(\nabla\vec{\phi})^2 + \underbrace{\frac{1}{2}r_0\vec{\phi}^2 + \frac{1}{4!}u_0\phi^4}_{V[\phi]}$$

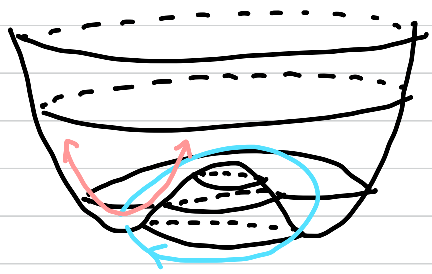
Minimum of $V[\phi]$

$$\frac{\partial V}{\partial \phi} = 0 = \left(r_0\vec{\phi} + \frac{1}{6}u_0\vec{\phi}^2\vec{\phi}\right)$$

$$\Rightarrow \vec{\phi}^2 = -6\frac{r_0}{u_0} \quad \text{or} \quad 0$$

If $r_0 > 0 \Rightarrow$ minimum at $\vec{\phi} = 0$
If $r_0 < 0 \Rightarrow$ minimum at $\vec{\phi}^2 = -\frac{6r_0}{u_0}$: infinitely degenerated



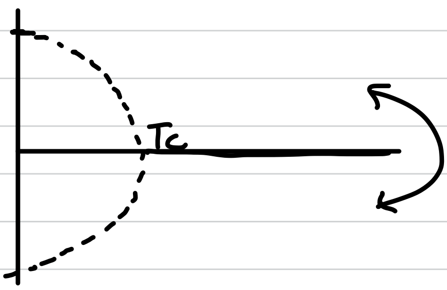


Mexican hat

IX.3. Classical Spontaneous Symmetry Breaking

For $h=0$, Ising model

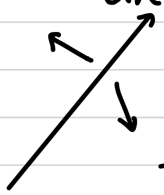
$$H = -\frac{J}{k_BT}\sum_{\langle ij\rangle} s_i s_j \quad : \mathbb{Z}_2 \text{ symmetry } \; s_i \to -s_i$$



$h \neq 0 \Rightarrow$ explicit breaking the symmetry

$\mathbb{Z}_2$ symmetry is not preserved near the ground state.
$\Rightarrow$ Spontaneous symmetry breaking (SSB)

direction of ordering $\hat{a}$



direction perpendicular to the ordering $\Rightarrow O(N-1)$ little group
$\leftrightarrow (N-1)$-modes

$(N-1)$-vector for $O(N)/O(N-1)$

$\Rightarrow S^{N-1}$ field space

$$\vec{\phi}^T(x) = (\sigma(x), \vec{\pi}(x))$$

$$S[\phi] = \frac{1}{2}\int d^dx \sum_{i,j=1}^{N} \delta_{ij} (\nabla \vec{\phi}^i)\cdot(\nabla \vec{\phi}^j)$$

$$\partial_\mu \sigma \partial^\mu \sigma + \sum_{a,b}^{N-1} \partial_\mu \pi^a \partial^\mu \pi^b \delta_{ab}$$

$$\left(\begin{array}{l} \partial_\mu \frac{1}{\sqrt{1-\pi^2}} \partial^\mu \frac{1}{\sqrt{1-\pi^2}} \\ \partial_\mu (1-\pi^2)^{\frac{1}{2}} = +\frac{1}{2} (1-\pi^2)^{-\frac{1}{2}} \times -2\partial_\mu \pi^a \\ \quad = \frac{-1}{\sqrt{1-\pi^2}} \pi_a \partial_\mu \pi^a \end{array} \right)$$

$$\Rightarrow S[\vec{\pi}] = \frac{1}{2}\int d^dx \sum_{a,b=1}^{N-1} \left(\delta_{ab} + \frac{\pi_a\pi_b}{1-\pi^2}\right)(\nabla\pi^a)\cdot(\nabla\pi^b)$$

$$\equiv \frac{1}{2}\int d^dx \sum_{a,b} g_{ab}(\vec{\pi})(\nabla\pi^a)\cdot(\nabla\pi^b)$$

where $g_{ab}(\vec{\pi}) = \delta_{ab} + \frac{\pi_a\pi_b}{1-\pi^2}$ is a <u>metric</u> on S^{N-1}.

At minimum : $\vec{\phi} = \hat{a}$

Propagator $G_{ab} = \frac{\delta_{ab}}{p^2} \Rightarrow$ No mass for $\vec{\pi}$.

In the classical (tree-level) analysis

- There exists infinitely degenerated ground state with symmetry $O(N-1)$
- $\vec{\pi}$ is massless. $\Rightarrow$ $N-1$ Goldstone modes exist.
- σ has no dynamics due to the constraint.
  $\Rightarrow$ σ mass becomes infinity.

IX.4. Ward-Takahashi Identities

$$\phi_i' = \sum_j D_{ij}(g)\phi_j \qquad g\in G$$ $\vec{\phi}$ is a vector w.r.t. G.

$\mathfrak{g} = \mathrm{Span}\{t_\alpha\}$

$$\delta\phi_i = \sum_\alpha \sum_j (t_\alpha)_{ij}\, \omega^\alpha \phi_j(x)$$

$$\delta S[\phi] = \sum_\alpha \sum_j (t_\alpha)_{ij} \frac{\delta S}{\delta\phi_i}\phi_j\omega^\alpha$$

If a given G is a symmetry of S then we demand $\delta S \overset{!}{=} 0$ for all ω_α $\Rightarrow$

$$\sum_j (t_\alpha)_{ij}\phi_j \frac{\delta S}{\delta \phi_i} \overset{!}{=} 0$$

$$\phi_i(x) = \phi_i'(x) + \sum_{j,\alpha}(t_\alpha)_{ij}\phi_j'(x)\omega_\alpha$$

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int dx\, \vec{J}(x)\cdot\vec{\phi}(x)}$$

$$\delta Z[J] \doteq 0 = \int \mathcal{D}\phi' \left[\int dx \sum_{ij\alpha} J_i(x)(t_\alpha)_{ij}\,\phi'_j(x)\,\omega_\alpha\right] e^{-S[\phi'] + \int dx\, \vec{J}\cdot\vec{\phi}'}$$

Here, we assume that the measure $\mathcal{D}\phi$ is is invariant under the transformation.
If not, the group is been breaking in quantum theory
$\Rightarrow$ anomaly

$$\Rightarrow 0 = \sum_\alpha \omega_\alpha \int \mathcal{D}\phi \int dx \sum_{i,j} J_i(x)\,(t_\alpha)_{ij}\,\underbrace{\phi_j(x)\, e^{-S[\phi] + \int dx \vec{J}\cdot\vec{\phi}}}_{\frac{\delta Z[J]}{\delta J_j}}$$

$$\Rightarrow \boxed{\begin{aligned} 0 &= \int dx \sum_{ij} (t_\alpha)_{ij}\, J_i \frac{\delta Z[J]}{\delta J_j} \\ &= \int dx \sum_{ij} (t_\alpha)_{ij}\, J_i \frac{\delta W[J]}{\delta J_j} \end{aligned}} \qquad W[J] = \log Z[J]$$

Consider the Legendre transformation

$$\Gamma[\varphi] + W[J] = \int dx\, \vec{J}(x)\cdot\vec{\varphi}(x) \qquad \varphi_i \equiv \frac{\delta W}{\delta J_i(x)}$$

$\hookrightarrow$ one particle irreducible (1PI) functional
"minimum free energy" $\quad W[0] \geq -\Gamma[\varphi]$

$$\ulcorner \log \langle e^{-J\cdot\phi}\rangle_J = W[0] - W[J] \geq \underbrace{\langle -J\,\phi\rangle_J}_{\Downarrow} = -J\cdot\varphi$$

$$\varphi(x) = \frac{\delta W[J]}{\delta J(x)} = \langle \phi(x)\rangle_J$$

$$\Rightarrow W[0] \geq -\Gamma[\varphi] \rightsquigarrow \text{variational principle.}$$

$\lrcorner$

$$\Rightarrow \boxed{\int dx \sum_{i,j} (t_\alpha)_{ij}\, \varphi_i(x) \frac{\delta\Gamma}{\delta\varphi_j(x)} = 0}$$

IX.5. Ward – Takahashi for O(N) Model

Transformation of ϕ under $O(N)/O(N-1)$

$$\begin{cases} \delta\pi_a = \omega_a \sigma \\ \delta\sigma = -\vec{\omega}\cdot\vec{\pi} \end{cases} \qquad Z[\vec{J}] = Z[\vec{J},K]\Big|_{K=k}.$$

$$Z[\vec{J},K] = \int \frac{\mathcal{D}\vec{\pi}}{\sqrt{1-\vec{\pi}^2}}\, e^{\frac{1}{g}\left[-S[\vec{\pi}] + \int d^dx\,(K(x)\sigma(x) + \vec{J}(x)\cdot\vec{\pi}(x))\right]}$$

$$\begin{cases} \delta\int d^dx\, \vec{J}(x)\cdot\vec{\pi}(x) = \int d^dx\, \vec{\omega}\cdot\vec{J}(x)\,\sigma(x) \\ \delta\int d^dx\, K(x)\,\sigma(x) = -\int d^dx\, K(x)\,\vec{\omega}\cdot\vec{\pi}(x) \end{cases}$$

Performing the infinitesimal transformation

$$0 = \int \frac{\mathcal{D}\vec{\pi}}{\sqrt{1-\vec{\pi}^2}} \int d^dx \left[\sigma(x) \vec{J}(x) - K(x) \vec{\pi}(x) \right] e^{\frac{1}{g}\left(-S[\vec{\pi}] + \int d^dx \{ K\sigma + \vec{J}\cdot\vec{\pi} \}\right)}$$

$$\Rightarrow \boxed{\int d^dx \left(\vec{J}(x) \frac{\delta}{\delta K(x)} - K(x) \frac{\delta}{\delta \vec{J}(x)} \right) Z[\vec{J},K] = 0}$$ $W[\vec{J},K] = g \log Z[\vec{J},K]$

$$\Rightarrow \boxed{\int d^dx \left(\vec{J}(x) \frac{\delta}{\delta K(x)} - K(x) \frac{\delta}{\delta \vec{J}(x)} \right) W[\vec{J},K] = 0}$$

Legendre transformation

$$W[\vec{J},K] + \Gamma[\vec{\pi},K] = \int d^dx\, \vec{J}(x)\cdot\vec{\pi}(x), \quad \vec{\pi}(x) = \frac{\delta W}{\delta \vec{J}(x)}$$

$$\Leftrightarrow \frac{\delta W}{\delta K(x)}\Big|_{\vec{J}} = -\frac{\delta W}{\delta K(x)}\Big|_{\vec{\pi}}$$

$$\boxed{\int d^dx \left(\frac{\delta \Gamma}{\delta \vec{\pi}(x)} \frac{\delta \Gamma}{\delta K(x)} + H(x)\vec{\pi}(x) \right) = 0}$$ master equation

$$\Gamma[\vec{\pi},K] = S[\vec{\pi},K] + \mathcal{O}(g)$$

$$S[\vec{\pi},K] = S[\vec{\pi}] - \int d^dx\, H(x)\,\sigma(x)$$

+ Renormalisation

$$\int d^dx \left(\frac{\delta S_R}{\delta \vec{\pi}(x)} \frac{\delta S_R}{\delta K(x)} + K(x)\vec{\pi}(x) \right) = 0$$

$$S[\vec{\pi},H] = S_R[\vec{\pi}]$$